Math 302, Assignment 4

1. Suppose X, Y are two discrete RV's with joint p.m.f. according to the table below.

   (a) Calculate the marginal p.m.f. of X and of Y.

   (b) Calculate $\mathbb{P}(0 < \sin(X) \cdot e^Y < 4)$.

   (c) Are X and Y independent?

   (d) Compute $\text{cov}(X, Y)$.

Table 1: The joint p.m.f. of X, Y

$X \downarrow Y \rightarrow$	0	1	2	3
1/2	1/12	1/8	1/8	1/12
1	0	1/12	1/9	1/9
6	1/12	1/12	0	1/9

Solution: (a) We have

$$p_X(1/2) = 5/12,\ p_X(1) = 11/36,\ p_X(6) = 5/18,$$

and

$$p_Y(0) = 1/6,\ p_Y(1) = 7/24,\ p_Y(2) = 17/72,\ p_Y(3) = 11/36.$$

(b) We have

$$\begin{aligned}\mathbb{P}(0 < \sin(X) \cdot e^Y < 4) &= \mathbb{P}((X, Y) \in \{(1/2, 0), (1/2, 1), (1/2, 2), (1, 0), (1, 1)\}) \\ &= \frac{1}{12} + \frac{1}{8} + \frac{1}{8} + \frac{1}{12} = \frac{5}{12}.\end{aligned}$$

(c) Since the p.m.f's are never zero, if the RV's were independent, the joint p.m.f. would never be zero either. But since there are two zeros in the table, this is not the case. Thus, the RV's are dependent.

(d) According to the expression for the marginals, the expectations are $\mathbb{E}X = \frac{11}{36} + \frac{30}{18} = \frac{71}{36}$ and $\mathbb{E}Y = \frac{7}{24} + \frac{34}{72} + \frac{33}{36} = \frac{121}{72}$. The expectation $\mathbb{E}XY = \frac{1}{16} + \frac{1}{12} + \frac{1}{2} + \frac{1}{8} + \frac{2}{9} + \frac{3}{24} + \frac{1}{3} + \frac{18}{9} = \frac{497}{144}$, and so $\text{cov}(X, Y) = -\frac{255}{2592}$

2.* Let Z_1 and Z_2 be two points chosen uniformly from the unit disk, independently of each other. Let $d(Z_1, Z_2)$ be their Euclidean distance, that is, if $z_i = (x_i, y_i)$, then $d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Compute $\mathbb{E}(d(Z_1, Z_2)^2)$.

Solution: Since the points are chosen uniformly, the p.d.f. of Z_i, $i = 1, 2$ is

$$f_i(x, y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

By independence, the joint p.d.f. of (Z_1, Z_2) is

$$f((x_1,y_1),(x_2,y_2)) = \begin{cases} \frac{1}{\pi^2} & x_1^2+y_1^2 \le 1 \text{ and } x_2^2+y_2^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We compute

$$\begin{aligned}
\mathbb{E}(d(Z_1,Z_2)^2) &= \frac{1}{\pi^2} \iint_{x_1^2+y_1^2\le 1} \iint_{x_2^2+y_2^2\le 1} \left[(x_1-x_2)^2 + (y_1-y_2)^2\right] \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \\
&= \frac{1}{\pi^2} \iint_{x_1^2+y_1^2\le 1} \iint_{x_2^2+y_2^2\le 1} \left[x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2x_1x_2 - 2y_1y_2\right] \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \\
&= \frac{4}{\pi^2} \iint_{x_1^2+y_1^2\le 1} \iint_{x_2^2+y_2^2\le 1} x_1^2 \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2,
\end{aligned}$$

where we have used symmetry. We calculate

$$\begin{aligned}
\iint_{x_1^2+y_1^2\le 1} \iint_{x_2^2+y_2^2\le 1} x_1^2 \, \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 &= \pi \iint_{x_1^2+y_1^2\le 1} x_1^2 \, \mathrm{d}x_1 \mathrm{d}y_1 \\
&= \pi \int_{-1}^{1} x_1^2 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} dy_1 \, dx_1 \\
&= \pi \int_{-1}^{1} 2x_1^2 \sqrt{1-x_1^2} = \frac{\pi^2}{4}
\end{aligned}$$

and conclude $\mathbb{E}(d(Z_1,Z_2)^2) = 1$.
Second solution: Denote $\mathbb{E}(d(Z_1,Z_2)^2) = \mathbb{E}(X_1^2 + X_2^2 + Y_1^2 + Y_2^2 - 2X_1X_2 - 2Y_1Y_2) = 4\mathbb{E}(X_1^2)$, by symmetry, and

$$\mathbb{E}(X_1^2) = \int x_1^2 f_{X_1}(x_1)\, dx_1 = \frac{1}{\pi} \int_{-1}^{1} 2x_1^2 \sqrt{1-x_1^2}\, dx_1 = \frac{1}{4}$$

3. Suppose that $X_1, \ldots, X_n$ are independent continuous random variables that all have the same c.d.f. $F(x)$. Define the random variable

$$Y = \max\{X_1, \ldots, X_n\}.$$

Compute the c.d.f. and the p.d.f. of Y. Your answer should be in terms of $F(x)$. *Hint:* Express an inequality of the kind $\max\{X_1, \ldots, X_n\} \le b$ in terms of separate inequalities for each X_i.

Solution: Since $\max\{X_1, \ldots, X_n\} \le x$ is equivalent to $\{X_1 \le x\} \cap \cdots \cap \{X_n \le x\}$, we can use independence and the definition of Y to obtain

$$\begin{aligned}
F_Y(x) &= \mathbb{P}(Y \le x) \\
&= \mathbb{P}(X_1 \le x, \ldots, X_n \le x) \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i \le x) \\
&= (F(x))^n.
\end{aligned}$$

We can now get the p.d.f. by differentiation. The chain rule gives

$$f_Y(x) = F_Y'(x) = nF(x)^{n-1}F'(x) = nF(x)^{n-1}f(x).$$

An equivalent approach would have been to use the "transformation of random variables" philosophy. The joint p.d.f. of $X_1, \dots, X_n$ is the n variable function $f(x_1) \cdot f(x_2) \cdots f(x_n)$, and we can find the c.d.f. of $\max\{X_1, \dots, X_n\}$, we have to integrate this function over the set $\max\{x_1, \dots, x_n\} \leq b$, and get

$$F_Y(x) = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} f(x_1) \cdots f(x_n)\ dx_1 \cdots dx_n,$$

which equals $F(x)^n$ as before.

4. Let X and Y be two independent uniform random variables on $(0, 1)$.
   (a) Using the convolution formula, find the p.d.f. $f_Z(z)$ of the random variable $Z = X + Y$, and graph it.
   (b) What is the moment generating function of Z?

Solutions: (a) The convolution formula gives us

$$f_Z(z) = \int_{-\infty}^{\infty} f(x)f(z - x)dx$$

and note that the interval for which the integrand is nonzero depends on z since we must have $0 < x < 1$ and $0 < z - x < 1$, or equivalently:

$$0 < x < 1 \text{ and } z - 1 < x < z.$$

Thus we integrate on the intersection interval $I = (0, 1) \cap (z - 1, z)$, that is, we have

$$f_Z(z) = \int_I 1\,dx.$$

First case: If $z \leq 0$ or $z \geq 2$ then $I = \emptyset$, so $f_Z(z) = 0$.
Second case: If $0 < z < 1$ then $I = (0, z)$, and $f_Z(z) = \int_0^z 1\,dx = z$.
Third case: If $1 \leq z < 2$ then $I = (z - 1, 1)$, and $f_Z(z) = \int_{z-1}^1 1\,dx = 2 - z$.
The graph of f_Z looks like a triangle whose three vertices have coordinates $(0, 0)$, $(1, 1)$ and $(2, 0)$.

(b) Let $M(t)$ be the moment-generating function, then by independence of X and Y,

$$M(t) = M_X(t)M_Y(t) = M_X(t)^2$$

since X and Y are both uniform on $(0, 1)$. On the other hand, $M_X(t) = \int_0^1 e^{tx}dx = e^{tx}/t|_0^1 = (e^t - 1)/t$. Thus,

$$M_Z(t) = \frac{(e^t - 1)^2}{t^2}$$