The University of British Columbia

Math 302 — Introduction to Probability

2017, December 8

Final Exam

Name: ______________________________ Student ID: ____________________

Instructions

- This exam consists of **8 questions** worth a total of 100 points.
- Make sure this exam has **13 pages** excluding this cover page.
- Note that there are tables on the last two pages.
- **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so). No credit might be given for unsupported answers.
- No calculators, notes, or other aids are allowed.
- If you need more space, use the back of the pages.
- Duration: **150** minutes.

Question	Points	Score
1	9	
2	11	
3	12	
4	9	
5	10	
6	17	
7	20	
8	12	
Total:	100	

1. New drugs against agoraphobia are tested on 300 people. 100 get drug A, which is effective 50% of the time. 100 get B which is effective 20% of the time, and 100 get the placebo, which is effective 30% of the time. Individuals are not told which group they are in.

4 marks (a) If a person in the study gets better, what is the probability that he was getting drug A?

Define the events

$$\begin{aligned} A &= \text{the person in the study gets better,} \\ B_1 &= \text{the person gets drug A,} \\ B_2 &= \text{the person gets drug B,} \\ B_3 &= \text{the person gets the placebo.} \end{aligned}$$

We know that $\mathbb{P}(B_1) = \mathbb{P}(B_2) = \mathbb{P}(B_3) = 1/3$, and $\mathbb{P}(A \,|\, B_1) = 0.5$, $\mathbb{P}(A \,|\, B_2) = 0.2$, and $\mathbb{P}(A \,|\, B_3) = 0.3$.
By Bayes' theorem the conditional probability in question is

$$\mathbb{P}(B_1 \,|\, A) = \frac{\mathbb{P}(B_1)\mathbb{P}(A \,|\, B_1)}{\sum_{i=1}^{3} \mathbb{P}(B_i)\mathbb{P}(A \,|\, B_i)} = \frac{1/3 \cdot 0.5}{1/3(0.5 + 0.2 + 0.3)} = \frac{1}{2}.$$

5 marks (b) If a person in the study does not get better, what is the probability that he was getting the placebo?

Bayes' theorem and $\mathbb{P}(\overline{A} \,|\, B_i) = 1 - \mathbb{P}(A \,|\, B_i)$ imply that the conditional probability in question is

$$\mathbb{P}\left(B_3 \,|\, \overline{A}\right) = \frac{\mathbb{P}(B_3)\mathbb{P}\left(\overline{A} \,|\, B_3\right)}{\sum_{i=1}^{3} \mathbb{P}(B_i)\mathbb{P}\left(\overline{A} \,|\, B_i\right)} = \frac{1/3 \cdot 0.7}{1/3(0.5 + 0.8 + 0.7)} = \frac{7}{20}.$$

2. Two players toss one 5-sided die each. The winner is the player with the larger result, with a tie in case of equal tosses.

3 marks (a) Write down a sample space for this experiment.

$$S = \{(1,1), \dots, (1,5), (2,1), \dots, (5,5)\}$$

4 marks (b) What is the probability of player 1 winning if they toss a 3?

Define the events

$$A = \text{the first player wins,}$$
$$B = \text{the first player tosses a 3.}$$

As $A \cap B$ consists of 2 pairs $(3,1), (3,2)$, the conditional probability is

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{2/25}{1/5} = \frac{2}{5}.$$

4 marks (c) What is the probability that player 1 threw a 3 if they won?

Since A consists of $\binom{5}{2} = 10$ pairs, the conditional probability is

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{2/25}{10/25} = \frac{1}{5}.$$

3 marks 3. (a) Define carefully what it means for two events A and B to be independent.

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

3 marks (b) Define carefully what it means for two events B and C to be mutually exclusive.

$B \cap C = \emptyset$

6 marks (c) Let A, B, C be events such that A and B are independent, and B and C are mutually exclusive. Assume that $\mathbb{P}(B) = \frac{1}{2}$, $\mathbb{P}(A \cap B) = \frac{1}{4}$, and $\mathbb{P}(C \cap (A \setminus B)) = \frac{1}{4}$. Compute $\mathbb{P}(C|A)$.

Since $C \cap B = \emptyset$, we have $\mathbb{P}(C \cap A) = \mathbb{P}(C \cap (A \setminus B)) = \frac{1}{4}$. Further, since A and B are independent, we have $\mathbb{P}(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{2}$. Therefore, $\mathbb{P}(C|A) = \frac{1}{2}$.

4. A group of 5 students are chosen at random from a class of 15 math and 20 engineering students.

4 marks (a) What is the probability that exactly 2 math students are chosen?

Preferable outcomes: $\binom{15}{2}\binom{20}{3}$, all otucomes: $\binom{35}{5}$, so

$$P = \frac{\binom{15}{2}\binom{20}{3}}{\binom{35}{5}}.$$

5 marks (b) What is the expected number of math students that are chosen? (Simplify your answer.)

Let $X_i = 1$ if the ith math student is chosen and 0 otherwise for $1 \leq i \leq 15$. Then the number of math students is $X = X_1 + \cdots + X_{15}$, and $\mathbb{P}(X_i = 1) = \frac{15}{35} = \frac{3}{7}$ for all i. Using the linearity of expectation

$$\mathbb{E}(X) = \sum_{i=1}^{15} \mathbb{E}(X_i) = 15\mathbb{E}(X_1) = \frac{45}{7}.$$

10 marks 5. Let X be a random variable with moment generating function

$$M_X(t) = (1-2t)^{-3/2} \quad \text{for } t < 1/2.$$

Calculate the mean and variance of X.

We have $M'(t) = 3(1-2t)^{-5/2}$ and $M''(t) = 15(1-2t)^{-7/2}$. Therefore,

$$\mathbb{E}(X) = M'(0) = 3, \quad \mathbb{E}(X^2) = M''(0) = 15,$$

and

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 15 - 3^2 = 6.$$

6. Assume that the random variables X, Y are jointly continuous with density function

$$f(x,y) = \begin{cases} c\frac{y}{x^2+y^2} & x > 0 \text{ and } 0 < y < 2 \\ 0 & \text{else} \end{cases}$$

3 marks

(a) Define carefully the normalization condition that the joint p.d.f. $f(x,y)$ has to satisfy.

$$\iint_{\mathbb{R}^2} f(x,y)\ dxdy = 1$$

5 marks

(b) What is the value of c? *Hint:* $\int \frac{1}{x^2+y^2}\ dx = \frac{1}{y} \arctan \frac{x}{y}$

$$c^{-1} = \int_0^2 \int_0^\infty \frac{y}{x^2+y^2} = \int_0^2 \frac{\pi}{2} = \pi$$

5 marks

(c) Find the conditional p.d.f. of X given $Y = 1$.

We calculate the marginal

$$f_Y(y) = \int_0^\infty \tfrac{1}{\pi} \frac{y}{x^2 + y^2} \, dx = \tfrac{1}{2}$$

and conclude

$$f_{X|Y}(x|1) = \begin{cases} \frac{2}{\pi} \frac{1}{1+x^2} & x > 0 \\ 0 & \text{else} \end{cases}$$

4 marks

(d) Find the conditional expectation $\mathbb{E}(X|Y = 1)$

$$\mathbb{E}(X|Y = 1) = \int_0^\infty \tfrac{2}{\pi} \frac{x}{1 + x^2} = \infty$$

7. Let (X, Y) be jointly continuous random variables with joint p.d.f. $f(x, y)$.

3 marks (a) Define the covariance of X and Y in terms of integrals of f.

$$\mathrm{Cov}(X,Y) = \iint_{\mathbb{R}^2} xyf(x,y)\ dxdy - \Big(\iint_{\mathbb{R}^2} xf(x,y)\ dxdy\Big)\Big(\iint_{\mathbb{R}^2} yf(x,y)\ dxdy\Big)$$

4 marks (b) Give an example of two random variables (X, Y) with $\mathrm{Cov}(X, Y) = 1$.

The pair (X, X) works for any RV X with variance 1. For a jointly continuous pair, see (c) below.

7 marks

(c) Assume that (X, Y) are uniform in the region $A = \{0 \leq x \leq 1, 0 \leq y \leq 2 \cdot a \cdot x\}$, where $a > 0$. Assume further that $\text{Cov}(X, Y) = \frac{1}{3}$. Find a.

The area of A is a, and we compute

$$\begin{aligned}
\text{Cov}(X,Y) &= \int_0^1 \int_0^{2ax} \frac{xy}{a} \, dydx - \int_0^1 \int_0^{2ax} \frac{x}{a} \, dydx \cdot \int_0^1 \int_0^{2ax} \frac{y}{a} \, dydx \\
&= \int_0^1 \frac{x}{2a} 4a^2 x^2 \, dx - \int_0^1 2x^2 \, dx \cdot \int_0^1 \frac{1}{2a} 4a^2 x^2 \\
&= \frac{a}{2} - \frac{4a}{9} = \frac{a}{18}
\end{aligned}$$

Therefore, $a = 6$.

6 marks (d) For (X, Y) uniform as in part (c), compute the p.d.f. of the random variable $\frac{X}{Y}$

For $b < 0$, we have $\mathbb{P}(\frac{X}{Y} \leq b) = 0$ since X and Y are both positive. For $0 \leq b \leq \frac{1}{12}$, we have $\mathbb{P}(\frac{X}{Y} \leq b) = \mathbb{P}(Y \geq \frac{X}{b}) = 0$. Finally, for $b > \frac{1}{12}$,

$$\mathbb{P}(\tfrac{X}{Y} \leq b) = \int_0^1 \int_{\frac{x}{b}}^{12x} \frac{1}{6} = \frac{1}{6} \int_0^1 (12 - \tfrac{1}{b})x = 1 - \tfrac{1}{12b}$$

and so

$$f_{\frac{X}{Y}}(b) = \begin{cases} \frac{1}{12b^2} & b > \frac{1}{12} \\ 0 & b < \frac{1}{12} \end{cases}$$

8. Let X be a $\text{Bin}(n, p)$ random variable.

3 marks (a) State the Poisson approximation for $\mathbb{P}(X = k)$.

$$\mathbb{P}(X = k) \approx \frac{\lambda^k}{k!}e^{-\lambda}$$

for $\lambda = n \cdot p$. This can be used for n large and p small.

4 marks (b) State the central limit theorem for X.

$$\mathbb{P}\left(\frac{X - n\cdot p}{\sqrt{np(1-p)}} \leq x\right) \to \mathbb{P}(\mathcal{N}(0,1) \leq x)$$

5 marks (c) Let $X \sim \text{Bin}(n, p)$ have mean 500 and variance 100. Approximate $\mathbb{P}(X \leq 520)$ using the central limit theorem.

$$\mathbb{P}(X \leq 520) \approx \mathbb{P}(Z \leq \tfrac{520-500}{\sqrt{100}}) = \mathbb{P}(Z \leq 2) = 0.97725.$$

Common Distributions

Random Variable X	p.m.f./p.d.f.	Mean	Variance
Ber(p)	$P(X=0)=1-p$, $P(X=1)=p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
Geom(p)	$p(1-p)^{k-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$P(X=r+k)=\binom{r+k-1}{r-1}p^r(1-p)^k$	$\frac{r}{p}$	$r\frac{1-p}{p^2}$
Poiss(λ)	$\frac{\lambda^k}{k!}e^{-\lambda}$	λ	λ
Unif$([a,b])$	$\frac{1}{b-a}$ if $x\in[a,b]$, 0 else	$\frac{a+b}{2}$	$\frac{1}{12}(b-a)^2$
Exp(λ)	$\lambda e^{-\lambda x}$ if $x>0$, 0 else	λ^{-1}	λ^{-2}
$\mathcal{N}(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2

Table 1: Values of $\Phi(z) = \int_{-\infty}^{z} f(x)dx$ where $f(x) = (2\pi)^{-1/2}e^{-x^2/2}$ is the density of an $\mathcal{N}(0,1)$ random variable.

z	0	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0	0,5	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,5279	0,53188	0,53586
0,1	0,53983	0,5438	0,54776	0,55172	0,55567	0,55962	0,56356	0,56749	0,57142	0,57535
0,2	0,57926	0,58317	0,58706	0,59095	0,59483	0,59871	0,60257	0,60642	0,61026	0,61409
0,3	0,61791	0,62172	0,62552	0,6293	0,63307	0,63683	0,64058	0,64431	0,64803	0,65173
0,4	0,65542	0,6591	0,66276	0,6664	0,67003	0,67364	0,67724	0,68082	0,68439	0,68793
0,5	0,69146	0,69497	0,69847	0,70194	0,7054	0,70884	0,71226	0,71566	0,71904	0,7224
0,6	0,72575	0,72907	0,73237	0,73565	0,73891	0,74215	0,74537	0,74857	0,75175	0,7549
0,7	0,75804	0,76115	0,76424	0,7673	0,77035	0,77337	0,77637	0,77935	0,7823	0,78524
0,8	0,78814	0,79103	0,79389	0,79673	0,79955	0,80234	0,80511	0,80785	0,81057	0,81327
0,9	0,81594	0,81859	0,82121	0,82381	0,82639	0,82894	0,83147	0,83398	0,83646	0,83891
1	0,84134	0,84375	0,84614	0,84849	0,85083	0,85314	0,85543	0,85769	0,85993	0,86214
1,1	0,86433	0,8665	0,86864	0,87076	0,87286	0,87493	0,87698	0,879	0,881	0,88298
1,2	0,88493	0,88686	0,88877	0,89065	0,89251	0,89435	0,89617	0,89796	0,89973	0,90147
1,3	0,9032	0,9049	0,90658	0,90824	0,90988	0,91149	0,91309	0,91466	0,91621	0,91774
1,4	0,91924	0,92073	0,9222	0,92364	0,92507	0,92647	0,92785	0,92922	0,93056	0,93189
1,5	0,93319	0,93448	0,93574	0,93699	0,93822	0,93943	0,94062	0,94179	0,94295	0,94408
1,6	0,9452	0,9463	0,94738	0,94845	0,9495	0,95053	0,95154	0,95254	0,95352	0,95449
1,7	0,95543	0,95637	0,95728	0,95818	0,95907	0,95994	0,9608	0,96164	0,96246	0,96327
1,8	0,96407	0,96485	0,96562	0,96638	0,96712	0,96784	0,96856	0,96926	0,96995	0,97062
1,9	0,97128	0,97193	0,97257	0,9732	0,97381	0,97441	0,975	0,97558	0,97615	0,9767
2	0,97725	0,97778	0,97831	0,97882	0,97932	0,97982	0,9803	0,98077	0,98124	0,98169
2,1	0,98214	0,98257	0,983	0,98341	0,98382	0,98422	0,98461	0,985	0,98537	0,98574
2,2	0,9861	0,98645	0,98679	0,98713	0,98745	0,98778	0,98809	0,9884	0,9887	0,98899
2,3	0,98928	0,98956	0,98983	0,9901	0,99036	0,99061	0,99086	0,99111	0,99134	0,99158
2,4	0,9918	0,99202	0,99224	0,99245	0,99266	0,99286	0,99305	0,99324	0,99343	0,99361
2,5	0,99379	0,99396	0,99413	0,9943	0,99446	0,99461	0,99477	0,99492	0,99506	0,9952
2,6	0,99534	0,99547	0,9956	0,99573	0,99585	0,99598	0,99609	0,99621	0,99632	0,99643
2,7	0,99653	0,99664	0,99674	0,99683	0,99693	0,99702	0,99711	0,9972	0,99728	0,99736
2,8	0,99744	0,99752	0,9976	0,99767	0,99774	0,99781	0,99788	0,99795	0,99801	0,99807
2,9	0,99813	0,99819	0,99825	0,99831	0,99836	0,99841	0,99846	0,99851	0,99856	0,99861
3	0,99865	0,99869	0,99874	0,99878	0,99882	0,99886	0,99889	0,99893	0,99896	0,999
3,1	0,99903	0,99906	0,9991	0,99913	0,99916	0,99918	0,99921	0,99924	0,99926	0,99929
3,2	0,99931	0,99934	0,99936	0,99938	0,9994	0,99942	0,99944	0,99946	0,99948	0,9995
3,3	0,99952	0,99953	0,99955	0,99957	0,99958	0,9996	0,99961	0,99962	0,99964	0,99965
3,4	0,99966	0,99968	0,99969	0,9997	0,99971	0,99972	0,99973	0,99974	0,99975	0,99976
3,5	0,99977	0,99978	0,99978	0,99979	0,9998	0,99981	0,99981	0,99982	0,99983	0,99983
3,6	0,99984	0,99985	0,99985	0,99986	0,99986	0,99987	0,99987	0,99988	0,99988	0,99989
3,7	0,99989	0,9999	0,9999	0,9999	0,99991	0,99991	0,99992	0,99992	0,99992	0,99992
3,8	0,99993	0,99993	0,99993	0,99994	0,99994	0,99994	0,99994	0,99995	0,99995	0,99995
3,9	0,99995	0,99995	0,99996	0,99996	0,99996	0,99996	0,99996	0,99996	0,99997	0,99997
4	0,99997	0,99997	0,99997	0,99997	0,99997	0,99997	0,99998	0,99998	0,99998	0,99998