The University of British Columbia

Final Examination - June 24, 2019

MATH 302

Closed book examination Time: 150 minutes

Last Name ______________________ **First** ______________________

Signature ______________ **Student Number** ______________________

Special Instructions:

No memory aids are allowed. No calculators may be used. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page and indicate that you have done so. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		20
3		10
4		10
5		10
6		10
Total		70

1. (10 pts) You roll a die 13 times. Let A be the event that you get exactly 2 sixes in your 13 rolls. Let B be the event that you get exactly two fives in your 13 rolls.

(a) What is $\mathbb{P}(A)$?

$X = \#$ of 6's $\sim \text{Bin}(13, 1/6)$

$$P(A) = P(X=2) = \boxed{\binom{13}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^{11}}$$

Answer

(b) What is $\mathbb{P}(A|B)$? *Do not simplify your solution.*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(B) = P(A)$ as above

$P(A \cap B) = P($ exactly 2 5's & exactly 2 6's $)$

$$= \binom{13}{9} \cdot \left(\frac{4}{6}\right)^9 \cdot \binom{4}{2} \left(\frac{1}{6}\right)^2 \cdot \left(\frac{1}{6}\right)^2$$

not 5 or 6 vals

or,

$$\frac{\# A \cap B}{\# \Omega} = \frac{\text{reordering of } a_1 a_2 \cdots a_9 5 5 6 6}{6^{13}}$$

where $a_i \in \{1, 2, 3, 4\}$

Answer

$$\frac{\binom{13}{9}\left(\frac{4}{6}\right)^9 \cdot \binom{4}{2} \frac{1}{6^4}}{\binom{13}{2} \cdot \left(\frac{5}{6}\right)^{11} \cdot \frac{1}{6^2}}$$

2. (20 pts) Suppose you are in line at Costco and there are 100 people in front of you. Assume that the time it takes each person in front of you to be served follows an Exponential distribution with expected value 4 (ignore units). Let X be the total time you wait for all 100 people in front of you to be served.

   (a) (2 pts) What is $\mathbb{E}[X]$?

$$X = \sum_{i=1}^{100} X_i \qquad X_i \overset{iid}{\sim} Exp(1/4)$$

$$\mathbb{E}X = \sum_{i=1}^{100} \mathbb{E}X_i = \sum_{i=1}^{100} 4 = \boxed{400}$$

Answer

   (b) (2 pts) What is $\text{Var}(X)$?

$$Var(X) = \sum_{i=1}^{100} Var(X_i) = \sum_{i=1}^{100} 4^2 = \boxed{1600}$$

$$\mathbb{E}\, Exp(\lambda) = \frac{1}{\lambda}$$

$$Var(Exp(\lambda)) = \frac{1}{\lambda^2}$$

Answer

(c) (4 pts) What is the moment generating function of X?

Hint: If $Z \sim Exp(\lambda)$

$$M_Z(t) = \begin{cases} \infty & t \geq \lambda \\ \frac{\lambda}{\lambda - t} & t < \lambda \end{cases}$$

$$M_X(t) = M_{X_1 + \dots + X_{100}}(t) = M_{X_1}(t) M_{X_2}(t) \cdots M_{X_{100}}(t)$$

$$= \left(M_{X_1}(t)\right)^{100} = \boxed{\begin{cases} \infty & t \geq \frac{1}{4} \\ \left(\frac{\frac{1}{4}}{\frac{1}{4} - t}\right)^{100} & t < \frac{1}{4} \end{cases}}$$

Answer

(d) (4 pts) Use Markov's inequality to bound $\mathbb{P}(X \geq 1000)$.

$$P(X \geq 1000) \leq \frac{EX}{1000} = \boxed{\frac{400}{1000}}$$

Answer

(e) (4 pts) Use Chebyshev's inequality to bound $\mathbb{P}(X \geq 1000)$.

$$P(X - 400 \geq 600) \leq P(|X - 400| \geq 600)$$

$$\underset{\text{Cheby.}}{\leq} \boxed{\frac{1600}{(600)^2}}$$

Answer

(f) (4 pts) Use a normal approximation to approximate $\mathbb{P}(X \geq 1000)$.

$$X \overset{dist}{\approx} N(400, 1600) = 400 + 40\,N(0,1)$$

so $\quad P(X \geq 1000) \approx P(400 + 40N(0,1) \geq 1000)$

$$= P\left(N(0,1) \geq \frac{\cancel{600}^{15}}{\cancel{40}}\right)$$

$$= \boxed{1 - \Phi(15)}$$

Answer

3. (10 pts) Let $X \sim N(0, 1)$. What is the probability density function of X^4?

$$Z = g(X) = X^4$$

$$f_Z(z) = \sum_{x : g(x) = z} \frac{f_X(x)}{|g'(x)|} = \sum_{x = z^{1/4}, -z^{1/4}} \frac{\phi(x)}{|4x^3|}$$

$$x : x^4 = z$$

$$= \frac{\phi(z^{1/4})}{4z^{3/4}} + \frac{\phi(z^{-\frac{1}{4}})}{|-4z^{3/4}|}$$

$$= \frac{2\,\phi(z^{1/4})}{4z^{3/4}} = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}\, z^{3/4}} e^{-\frac{z^{2/4}}{2}}$$

for $z \geq 0$.

For $z < 0$, $f_Z(z) = 0$

since $Z = X^4 \geq 0$

Answer

symmetry method

4. (10 pts) Let X and Y be independent Geometric(p) random variables.

(a) What is $\mathbb{P}(X < Y)$?

$$P(Y<X) + P(Y=X) + P(Y>X) = 1$$

by symmetry

$$\Rightarrow P(X<Y) = \frac{1-P(Y=X)}{2} \quad \ldots \quad (\text{left for reader})$$

direct method

$$P(X<Y) = \sum_{k=1}^{\infty} P(X<Y \mid X=k) \cdot P(X=k)$$

$$= \sum_{k=1}^{\infty} \underbrace{P(k<Y)}_{\text{1st } k \text{ trials fail}} \cdot P(X=k) = \sum_k (1-p)^k (1-p)^{k-1} p$$

$$= \sum_k (1-p)^{2(k-1)} (1-p) \cdot p = \frac{(1-p) \cdot p}{1-(1-p)^2} = \boxed{\frac{1-p}{2-p}}$$

Answer

(b) What is the probability mass function of the minimum $\min(X, Y)$?

$$Z = \min(X,Y)$$

$$P(Z>k) = P(\min(X,Y)>k)$$

$$= P(X>k, Y>k) = P(X>k) \cdot P(Y>k)$$

$$= (1-p)^k (1-p)^k$$

$$= (1-p)^{2k}$$

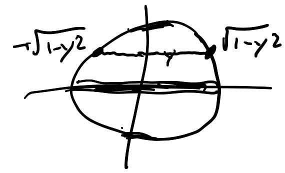
$$P(Z=k) + P(Z>k) = P(Z>k-1)$$

$$\text{so} \quad P(Z=k) = P(Z>k-1) - P(Z>k)$$

$$= \boxed{(1-p)^{2(k-1)} - (1-p)^{2k}}$$

Answer

5. Let the random vector (X, Y) be drawn uniformly from the disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$.



(a) (2 pts) What is the joint density function of (X, Y)?

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

Answer

(b) (4 pts) What is the marginal density function of Y?

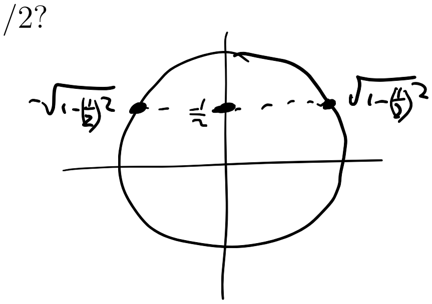
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}\, dx$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \quad \text{for } y \in [-1, 1]$$

$$f_Y(y) = 0 \quad \text{for } y \notin [-1, 1]$$

Answer

(c) (4 pts) What is the conditional density of X given $Y = 1/2$?



$$f_{X|Y}(x|\tfrac{1}{2}) = \frac{f_{X,Y}(x,y)}{f_Y(\frac{1}{2})}$$

$$= \left.\frac{\frac{1}{\pi}}{\frac{2\sqrt{1-y^2}}{\pi}}\right|_{y=\frac{1}{2}} = \left.\frac{1}{2\sqrt{1-y^2}}\right|_{y=\frac{1}{2}} = \frac{1}{2\sqrt{\frac{3}{4}}}$$

$$\text{for } x \in \left[-\sqrt{\tfrac{3}{4}}, \sqrt{\tfrac{3}{4}}\right]$$

Answer

$$f_{X|Y}(x|\tfrac{1}{2}) = 0 \quad \text{for } x \notin \left[-\sqrt{\tfrac{3}{4}}, \sqrt{\tfrac{3}{4}}\right]$$

6. (10 pts) A group of n people get on an elevator at Floor 0. There are m floors besides Floor 0. Each person uniformly randomly chooses one of those m floors to stop at. (All choices are independent, and no one gets on the elevator after Floor 0.) The elevator stops at a floor if at least one person has chosen to stop there. Let X be the number of stops that the elevator makes after Floor 0.

(a) What is $\mathbb{E}[X]$?

See review

Answer

(b) What is $\mathrm{Var}(X)$?

Answer

The End