The University of British Columbia
Math 302 — Introduction to Probability
2017, October 18

Name: ______________________________ Student ID: ____________________

Instructions

- This exam consists of **5 questions** worth a total of 40 points.
- Make sure this exam has **5 pages** excluding this cover page.
- Note that there is a **table of discrete distributions** on Page 1, too.
- **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so). No credit might be given for unsupported answers.
- It is stated in each problem **whether the answer should be simplified**. Simplified answers should be an expression like $\frac{34}{77}$, $\frac{2-p^2}{p+1}$, or $\frac{2}{p^2}+\frac{1}{p}$. Non-simplified answers may look like $\frac{2\binom{14}{7}+3\binom{15}{4}}{\binom{12}{5}}$, $\frac{4\cdot 3\cdot 5!}{8^4}$, or $\sum_{k=2}^{7} k\binom{15}{k}(0.8)^k(0.2)^{15-k}$.
- No calculators, notes, or other aids are allowed.
- If you need more space, use the back of the pages.
- Duration: **50** minutes.

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
Total:	40	

Common Discrete Distributions

Random Variable X	$P(X=k)$	Mean	Variance
$\mathrm{Ber}(p)$	$P(X=0)=1-p,\ P(X=1)=p$	p	$p(1-p)$
$\mathrm{Bin}(n,p)$	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
$\mathrm{Geom}(p)$	$p(1-p)^{k-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$\mathrm{NegBin}(r,p)$	$P(X=r+k)=\binom{r+k-1}{r-1}p^r(1-p)^k$	$\frac{r}{p}$	$r\frac{1-p}{p^2}$

8 marks

1. A cutlery set contains 6 knives and 6 forks. 4 pieces of cutlery are chosen at random. Find (do NOT simplify) the probability that:
   (a) 2 complete pairs (of one knife and one fork each) are chosen;
   (b) exactly 1 complete pair (knife + fork) is chosen

$\#\Omega = \binom{12}{4}$ $\qquad \# A \qquad A = \{2 \text{ forks}, 2 \text{ knives}\}$

a) 2 forks out of 6

$$\boxed{\frac{\#A}{\#\Omega} = \frac{\binom{6}{2}\cdot\binom{6}{2}}{\binom{12}{4}}}$$

2 knives out of 6

b) $B = \{\underbrace{1 \text{ fork } \& \text{ } 3 \text{ knives}}_{C} \text{ or } \underbrace{3 \text{ forks } \& \text{ } 1 \text{ knife}}_{D}\}$

$= C \cup D$

$$P(B) = P(C) + P(D) = 2P(C) = 2\frac{\#C}{\#\Omega}$$

$$= \boxed{\frac{2\binom{6}{1}\cdot\binom{6}{3}}{\binom{12}{4}}}$$

8 marks 2. Math 302 students decide to try to hack their instructor's PC to steal the midterm. If they succeed, they will all be able to score 100 on the test. Their chance of successfully completing the hack is 5%. Under normal conditions, only 10% of them would score 100.

(a) The instructor grades the first two tests, and finds that both of them score 100. What is the probability that his students have hacked his PC? Simplify your answer.

(b) Given this, what is the probability that the third test he grades will have score less than 100? Simplify your answer.

that 1st 2 tests score 100

a) $H = \{\text{hacked}\}$, $A = \{\text{both score } 100\}$

$$P(H|A) = \frac{P(A|H)\cdot P(H)}{P(A)} = \frac{1\cdot\frac{1}{20}}{P(A)}$$

$$P(A) = P(A|H)\cdot P(H) + P(A|H^c)\cdot P(H^c) = 1\cdot\frac{1}{20} + \frac{1}{10^2}\cdot\frac{19}{20}$$

Plug in above to give $P(H|A) = \dfrac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{10^2}\frac{19}{20}}$

b) $B = \{\text{3rd test} = 100\}$

$$P(B^c|A) = 1 - P(B|A)$$

$$P(B|A) = \frac{P(B,A)}{P(A)}$$

as above

$$P(B,A) = P(3\ 100\text{'s}) = P(B,A|H)\cdot P(H) + P(B,A|H^c)\cdot P(H^c)$$
$$= 1\cdot\frac{1}{20} + \frac{1}{10^3}\cdot\frac{19}{20}$$

Plug in above to finish.

8 marks 3. A hotel with 30 rooms makes reservations for 32 rooms, and it is known that on average 10% of people who reserve rooms do not show up. In this question, do NOT simplify your answer.

(a) What is the probability that there will be someone without a room?

(b) The hotel pays \$50 to any customer who shows up with a reservation but is denied a room due to overbooking. What is the expected amount that the hotel will have to pay out?

$X = \#$ who don't show up $\sim Bin(32, \frac{1}{10})$

a) $P(\text{missing a room}) = P(X=0 \text{ or } X=1) = P(0)+P(1)$

$= \binom{32}{0} \cdot \left(\frac{9}{10}\right)^{32} + \binom{32}{1} \cdot \frac{1}{10} \cdot \frac{9}{10}^{31}$

b) $\mathbb{E}(\text{payment}) = 100 \cdot P(X=0) + 50 \cdot P(X=1)$

$= 100 \cdot \binom{32}{0} \cdot \left(\frac{9}{10}\right)^{32} + 50 \cdot \binom{32}{1} \cdot \frac{1}{10} \cdot \left(\frac{9}{10}\right)^{31}$

8 marks

4. A company producing light bulbs claims that their product is *guaranteed to burn at least 1500 hours*. According to government regulations, this would be false advertisement if 4% or more of light bulbs were to burn out before 1500 hours. The government is suspicious and asks the company about statistics concerning how long their products actually last. The company replies that their light bulbs will burn on average 2'000 hours, with a standard deviation of 50 hours.

   (a) Show that the company has given enough information to prove that they are not guilty of false advertisement.

   (b) Keeping the mean fixed, how large can the standard deviation be at most to meet the governments regulation for truthful advertisement? Simplify your answer.

8 marks 5. Let X be a Geom(p) random variable, and $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = (x+1)^2$. Compute $\mathbb{E}\, g(X)$.

Recall: $\mathbb{E}X = \frac{1}{p}$ $\quad Var(X) = \frac{1-p}{p^2}$

so, $\mathbb{E}(X+1)^2 = \mathbb{E}[X^2 + 2X + 1] = \mathbb{E}[X^2] + 2\mathbb{E}X + \mathbb{E}1$

$$= \underbrace{\mathbb{E}[X^2] - (\mathbb{E}X)^2}_{Var(X)} + (\mathbb{E}X)^2 + 2\mathbb{E}X + \mathbb{E}1$$

$$= \frac{1-p}{p^2} + \frac{1}{p^2} + \frac{2}{p} + 1$$

$$\mathbb{E}g(X) = \sum_{k=1}^{\infty} (k+1)^2 \cdot (1-p)^{k-1} \cdot p = S$$

$$S = 4 \cdot p + \sum_{k=2}^{\infty} (k+1)^2 (1-p)^{k-1} \cdot p = 4p + \sum_{k=1}^{\infty} k^2 (1-p)^{k-2} p$$

$$= 4p + (1-p)^{-1} \sum_{k=1}^{\infty} \left((k+1)^2 - 2k - 1\right) (1-p)^{k-1} p$$

$$= 4p + (1-p)^{-1} \left[\underbrace{\sum_{k=1}^{\infty} (k+1)^2 (1-p)^{k-1} p}_{S} \underbrace{- 2 \sum_{k=1}^{\infty} k (1-p)^{k-1} p}_{-2\mathbb{E}X = -\frac{2}{p}} \underbrace{- \sum_{k=1}^{\infty} (1-p)^{k-1} p}_{-1} \right]$$

$$\Rightarrow S = 4p + \frac{1}{1-p}\left[S - \frac{2}{p} - 1\right]$$