The University of British Columbia

Midterm Examination - July 27, 2022

MATH 302: 951

Closed book examination Time: 50 minutes

Last Name ______________________ **First** ______________________

Signature ______________ **Student Number** ______________________

Special Instructions:

No memory aids are allowed. No calculators may be used. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page and indicate that you have done so. Please box your answers. You may use a 5 inch by 8 inch index card of notes (both sides).

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		10
4		10
5		10
Total		50

1. You have a bag with 10 green balls, 10 red balls, and 10 purple balls. You randomly draw 5 balls without replacement. Do not simplify your answers on this page.

   (a) What is the probability that all 5 of the drawn balls are green?

$A = \{\text{all 5 green}\}$ $\quad \# A = \binom{10}{5}$

$$P(A) = \frac{\# A}{\# \Omega} = \boxed{\frac{\binom{10}{5}}{\binom{30}{5}}}$$

   (b) What is the probability that at least 4 of the drawn balls are the same color?

$$P(\geq 4 \text{ of } 1 \text{ color}) = P(4 \text{ of } 1 \text{ color}) + P(5 \text{ of } 1 \text{ color})$$

$$= \boxed{\frac{\binom{3}{1} \cdot \binom{10}{4} \cdot \binom{20}{1}}{\binom{30}{5}} + \frac{\binom{3}{1} \cdot \binom{10}{5}}{\binom{30}{5}}}$$

2. Let Ω be a sample space with probability measure P. Consider events $A, B, C \in \Omega$ satisfying:
i) A and B are independent,
ii) A and C are disjoint,
iii) $P(A) = P(B) = P(C) = 1/3$,
iv) $P(A \cup B \cup C) = 2/3$.

since A, C are disjoint

(a) What is $P(A \cap B \cap C)$?

$$P(A \cap B \cap C) = P((A \cap C) \cap B) = P(\{\} \cap B)$$

$$= P(\{\}) = 0$$

(b) What is $P(B \cap C)$?

$$\frac{2}{3} = P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

inclusion exclusion

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3} - 0 - P(B \cap C) + 0$$

so $P(B \cap C) = 1 - \frac{1}{9} - \frac{2}{3} = \boxed{\frac{2}{9}}$

3. Let $X \sim Geom(p)$.

(a) What is the probability that X is odd given that $X \geq 8$?

$P(\text{odd} \mid X \geq 8) = P(X-7 \text{ is even} \mid X > 7)$

$= P(X \text{ is even})$

memoryless

$= p(2) + p(4) + p(6) + \dots$

$= (1-p)p + (1-p)^3 \cdot p + (1-p)^5 \cdot p + \dots$

$= \dfrac{(1-p)p}{1-(1-p)^2} = \dfrac{(1-p)\cdot p}{2p-p^2} = \boxed{\dfrac{1-p}{2-p}}$

.

(b) Let $Y \sim Unif[0, 3]$ be independent of X. What is the probability that $Y < X$?

$P(Y<X) = P(Y<X \mid X=1) \cdot P(X=1) + P(Y<X \mid X=2)\, P(X=2)$

$+ P(Y<X \mid X \geq 3) \cdot P(X \geq 3)$

$= P(Y<1) \cdot P(X=1) + P(Y<2) \cdot P(X=2) + 1 \cdot P(X \geq 3)$

$= \frac{1}{3} \cdot p + \frac{2}{3} \cdot p(1-p) + P(\text{1st 2 trials fail})$

$= \boxed{\frac{1}{3} \cdot p + \frac{2}{3} p(1-p) + (1-p)^2}$

4. Let X have cumulative distribution function given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x < -3 \\ (x+3)^3 & \text{if} \quad -3 \le x < -2 \\ 1 & \text{if} \quad x \ge -2. \end{cases}$$

(a) Is X continuous or discrete? Why?

$\boxed{\text{Continuous}}$ since its differentiable except at $x = -2$.

(b) If it is continuous, find the probability density function. If it is discrete, find the probability mass function.

$$f(x) = F'(x) = \begin{cases} 0 & x < -3 \\ 3(x+3)^2 & -3 \le x < -2 \\ 0 & x \geq -2 \end{cases}$$

(c) What is the expected value, EX? (*Hint: You may wish to change variables in the integral.*)

$$\mathbb{E}X = \int_{-\infty}^{\infty} f(x)\cdot x \, dx = \int_{-3}^{-2} 3(x+3)^2 \cdot x \, dx \qquad \text{set } y = x+3$$

$$= \int_0^1 3y^2(y-3)\, dy = \int_0^1 3y^3 - 9y^2 \, dy = \frac{3y^4}{4} - \frac{9y^3}{3}\Big|_0^1$$

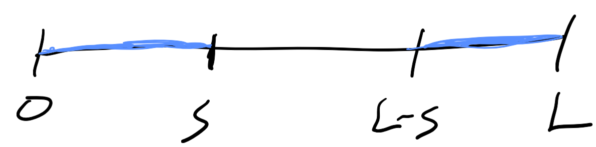
$$= \frac{3}{4} - 3 = \boxed{-2\tfrac{1}{4}}$$

5. A child has three sticks, which have lengths 1, 2, and 3 (ignore units). She chooses two sticks at random (without replacement). She breaks each of those two sticks at a uniform at random location. Then she takes the smallest of those 4 pieces. What is the probability that she chose the length 1 and length 2 sticks given that the piece she $\underbrace{\text{ends up with has length at least 1/10?}}_{\text{call this } A}$

Let $B_{i,j}$ be the event she chose sticks of lengths i & j

Consider a stick of length L, break it at uniform random location, & let X be the smaller piece.

Then $X \le s$ if the break is in the blue intervals.



Thus, $F(s) = P(X \le s) = \dfrac{s + L - (L-s)}{L} = \dfrac{2s}{L}$ for $0 \le s \le \frac{L}{2}$

so $X \sim Unif[0, \frac{L}{2}] = \frac{L}{2} Unif[0,1]$

Thus, letting $Y, Z \sim Unif[0,1]$ be indep.,

$P(A | B_{1,2}) = P(\min(\frac{1}{2}Y, Z) \ge \frac{1}{10}) = P(\frac{1}{2}Y \ge \frac{1}{10} \ \& \ Z \ge \frac{1}{10})$

$= P(Y \ge \frac{1}{5}) \cdot P(Z \ge \frac{1}{10}) = \frac{4}{5} \cdot \frac{9}{10}$, similarly, $P(A|B_{1,3}) = \frac{4}{5} \cdot \frac{14}{15}$

& $P(A|B_{2,3}) = \frac{9}{10} \cdot \frac{14}{15}$. Further, $P(B_{i,j}) = \dfrac{1}{\binom{3}{2}} = \frac{1}{3}$. Thus

$P(B_{1,2}|A) = \dfrac{P(A|B_{1,2}) \cdot P(B_{1,2})}{P(A)}$

$= \dfrac{P(A|B_{1,2}) \cdot \cancel{P(B_{1,2})}}{P(A|B_{1,2}) \cdot \cancel{P(B_{1,2})} + P(A|B_{1,3}) \cdot \cancel{P(B_{1,3})} + P(A|B_{2,3}) \cdot \cancel{P(B_{2,3})}} = \dfrac{\frac{4}{5} \cdot \frac{9}{10}}{\frac{4}{5} \cdot \frac{9}{10} + \frac{4}{5} \cdot \frac{14}{15} + \frac{9}{10} \cdot \frac{14}{15}}$

The End

$= \boxed{\dfrac{1}{1 + \frac{9}{10} \cdot \frac{14}{15} + \frac{4}{5} \cdot \frac{14}{15}}}$