

Lecture 7: Introduction to Partial Differential Equations (continued)

(Compiled 4 August 2017)

In this lecture we will continue with the derivation of the basic PDEs studied in this course from different physical situations. We deliberately explore the different paths to arrive at the same partial differential equations to emphasize way in which the models from disparate applications share the same features.

Key Concepts: Partial Differential Equations (PDEs); Elliptic, Parabolic, Hyperbolic PDEs; The heat Equation, The Wave Equation, and Laplace’s Equation, Modeling and Derivation of PDEs.

7 Introduction to PDEs (continued)

7.1 The Wave Equation:

Consider an elastic rod having a density ρ and cross-sectional area A , and let $\sigma(x, t)$ be the pressure in the rod at x at time t and $u(x, t)$ the displacement of the rod from its equilibrium position.

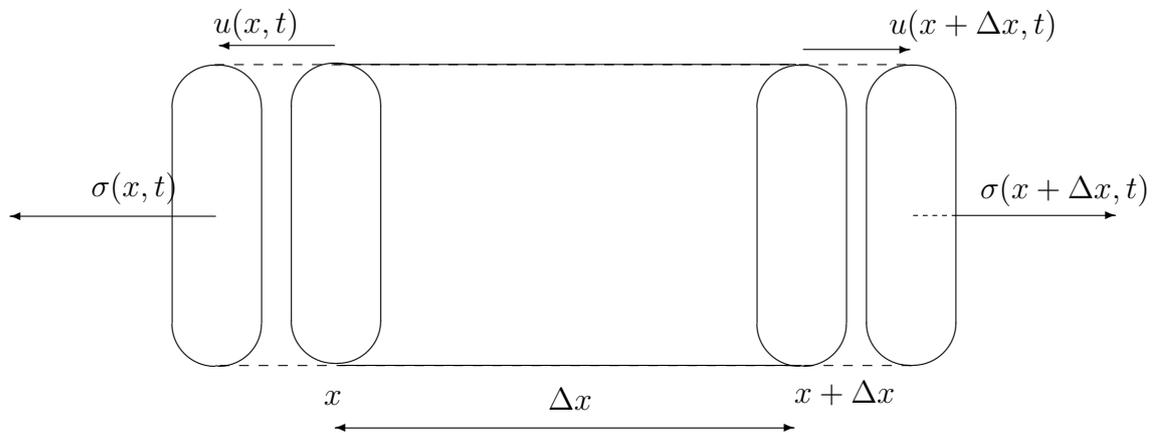


FIGURE 1. We consider the net force $\sigma(x + \Delta x, t)A - \sigma(x, t)A$ acting on the bar, which according to Newton’s Second Law of motion, must be balanced by the product of mass of the bar and its acceleration

Balance of Linear Momentum “ $F = Ma$ ”.

$$\begin{aligned} \sigma(x + \Delta x, t)A - \sigma(x, t)A &= \rho A \Delta x \frac{\partial^2 u}{\partial t^2} \\ \frac{\sigma(x + \Delta x, t) - \sigma(x, t)}{\Delta x} &= \rho \frac{\partial^2 u}{\partial t^2} \\ \text{Let } \Delta x \rightarrow 0, \text{ which yields } \frac{\partial \sigma}{\partial x} &= \rho \frac{\partial^2 u}{\partial t^2} \end{aligned} \tag{7.1}$$

We observe that (7.1) involves two unknown quantities the stress in the bar σ and the resulting displacement u . In

order to have sufficient information to solve for the unknowns we need an additional equation, which is provided by a constitutive relation known as Hooke's Law (see figure 2). Experimental data characterizes the "stiffness" of the material by the parameter E known as the Young's Modulus, which provides a linear relationship between the stress to which the bar is subjected and the relative displacement $\frac{\Delta u}{\Delta x} = \frac{u(x+\Delta x,t)-u(x,t)}{\Delta x} \approx \frac{\partial u}{\partial x} := \epsilon$, or strain ϵ .

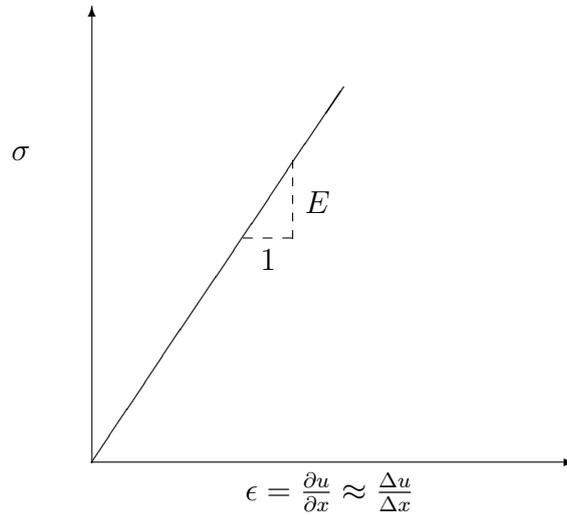


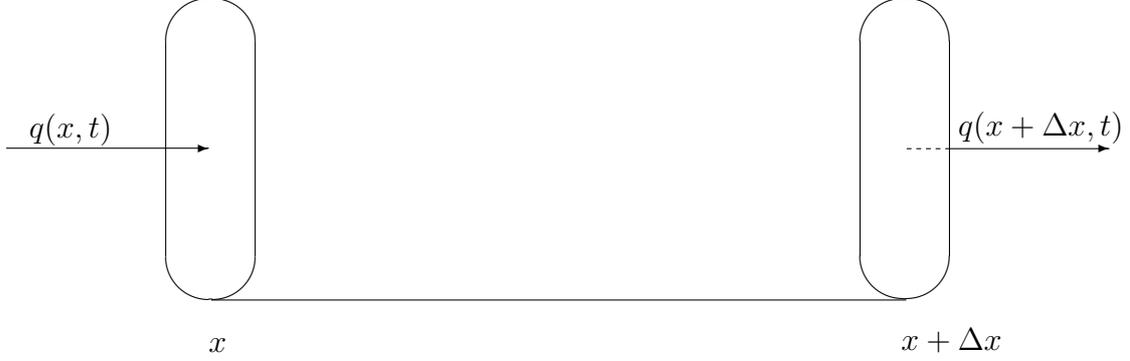
FIGURE 2. The stress on the bar σ is related to the strain ϵ by Hooke's Law

Substituting the stress strain relationship $\sigma = E \frac{\partial u}{\partial x}$ into (7.1) we obtain the second order wave equation.

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{E}{\rho} \right) \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ where } c = \sqrt{\frac{E}{\rho}} \quad (7.2)$$

Observation: We note that (7.2) is in precisely the same form as the wave equation that occurred in the context of modeling traffic flow down a highway. Equation (7.2) describes the motion of an acoustic wave that travels in an elastic bar.

7.2 The Heat/Diffusion Equation

FIGURE 3. Heat flux in a conducting bar that occupies the region $[x, x + \Delta x]$

7.2.1 Fourier's Law and Heat Conduction:

How do we build a model of the flow of heat in a conductor? Consider an elemental length of a conducting metal bar (think copper or aluminium) that occupies the interval $[x, x + \Delta x]$. We define the following material properties of the conductor:

Heat Capacity: Let C be the specific heat capacity of the material, which is defined to be the amount of heat required to change one kilogram of the material by one degree Kelvin, i.e., $[C] = \frac{J}{kgK}$.

Density: Let ρ be the density of the conductor, i.e., $[\rho] = \frac{kg}{m^3}$.

Heat Flux: Let q be the flux of heat energy per unit area, i.e., $[q] = \frac{J}{m^2s}$.

If A is the cross sectional area of the bar, then the change of temperature $u(x, t)$ within the element of length Δx over a time interval Δt is given by

$$C\Delta u\rho\Delta xA = C\{u(x, t + \Delta t) - u(x, t)\}\Delta xA \simeq \{q(x, t) - q(x + \Delta x, t)\}A\Delta t \quad (7.3)$$

Now dividing by $A\Delta x\Delta t$ and taking the limit as $\Delta x, \Delta t \rightarrow 0$, we obtain

$$\rho C \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (7.4)$$

It is our common experience that heat flows from hotter regions to cooler ones, which is illustrated in figure 4 in which the directions of the flux of heat q are indicated depending upon the sign of $\frac{\partial u}{\partial x}$. In addition, experimental evidence suggests that the flux of heat is proportional to the negative of the spatial gradient of the temperature, which relationship is known as Fourier's Law of heat conduction

$$q = -k \frac{\partial u}{\partial x} \quad (7.5)$$

where k is the thermal conductivity having dimensions $[k] = \frac{J}{smK}$. Substituting (7.5) into (7.3) and dividing by ρC we obtain the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad (7.6)$$

where $\alpha^2 = \frac{k}{\rho C}$ is the diffusion coefficient, which has dimensions $[\alpha^2] = \frac{m^2}{s}$.

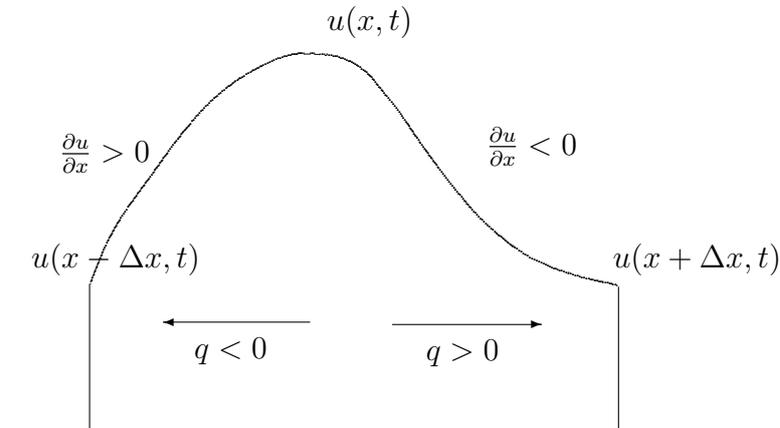


FIGURE 4. Fourier's Law of Heat Conduction: heat moves from points at which the temperature is higher in the direction of points at a lower temperature and the flux is given by $q = -k \frac{\partial u}{\partial x}$

A similar line of reasoning for the heat flow in a conduction plate leads to the two dimensional Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (7.7)$$

7.2.2 Fick's Law and diffusion:

Another way to arrive at the diffusion equation is to return to the conservation law derived to model traffic flow in the previous lecture:

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (7.8)$$

but instead of $u(x, t)$ representing the density of cars on a freeway, let us interpret u as the concentration of molecules of a certain chemical in a stream and q the flux of these molecules. In this case the constitutive relation between q and u is provided by Fick's law

$$q = -\alpha^2 \frac{\partial u}{\partial x} \quad (7.9)$$

Combining (7.9) with the conservation law (7.8) we recover (7.6), which in this context is known as the diffusion equation. That the same equation *aliases* as the heat equation or the diffusion equation stems from the distinct areas of application from which these names arise.

7.2.3 The drunkard's walk

Consider fruit flies having a density $u(x, t)$ at point x at time t located on a row of trees that are spaced Δx apart. We assume that the fruit flies will jump to the tree to the left with a probability p and to the right with the same probability p . The probability that the fruit flies stay on the tree is $1 - 2p$. Find an equation for the density of flies at $t + \Delta t$, i.e., $u(x, t + \Delta t)$.

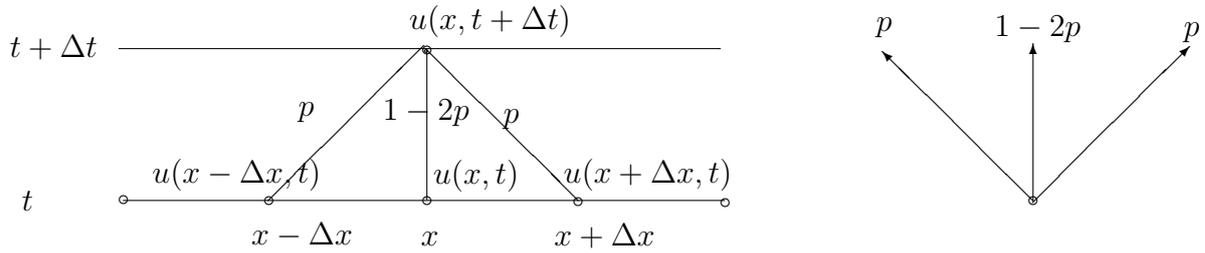


FIGURE 5. Consider fruit flies having a density $u(x, t)$ located on a row of trees that are spaced Δx apart

$$\begin{aligned}
 u(x, t + \Delta t) &= pu(x + \Delta x, t) + (1 - 2p)u(x, t) + pu(x - \Delta x, t) \\
 &= u(x, t) + p\Delta x \frac{[u(x + \Delta x, t) - u(x, t)]}{\Delta x} - \frac{[u(x, t) - u(x - \Delta x, t)]}{\Delta x} \\
 &\simeq u(x, t) + p\Delta x^2 \left\{ \frac{\partial u}{\partial x}(x, t) - \frac{\partial u}{\partial x}(x - \Delta x, t) \right\} \\
 &\simeq u(x, t) + p\Delta x^2 \frac{\partial^2 u}{\partial x^2} \\
 \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} &\simeq \left(p \frac{\Delta x^2}{\Delta t} \right) \frac{\partial^2 u}{\partial x^2} \tag{7.10}
 \end{aligned}$$

Now choose the mesh and time sampling such that when taking the limit $\Delta x, \Delta t \rightarrow 0$ we obtain the limiting value $\left(p \frac{\Delta x^2}{\Delta t} \right) \rightarrow \alpha^2$ so that (7.10) reduces to

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{The Heat Eq.} \tag{7.11}$$

Question: What is the Mean Absolute Deviation of a fruit fly?

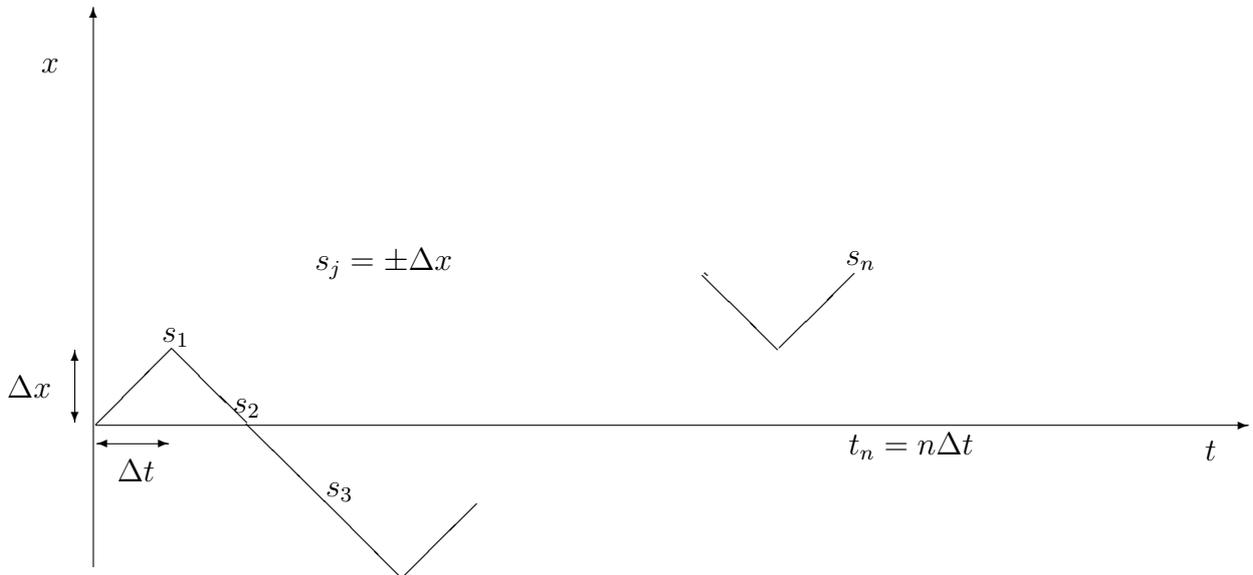


FIGURE 6. Consider the trajectory a single fruit fly in which it takes steps of size $\pm\Delta x$ each time step Δt . What is the mean deviation about its starting point?

$$\begin{aligned}
 s_j &= \pm\Delta x \\
 t_j &= j\Delta t
 \end{aligned}$$

$$x_N = s_1 + s_2 + \cdots + s_N \sim 0 \quad \text{Expected Value} \quad (7.12)$$

$$\begin{aligned} x_N^2 &= (s_1 + \cdots + s_N)^2 = s_1^2 + \cdots + s_N^2 + 2(s_1 s_2 + \cdots + s_{N-1} s_N) \\ &\sim N \Delta x^2 \end{aligned}$$

Therefore

$$\begin{aligned} x_N^2 &\simeq \left(\frac{t_N}{\Delta t} \right) \Delta x^2 = k^2 t_N \\ |x_N| &\sim k \sqrt{t_N} \end{aligned}$$

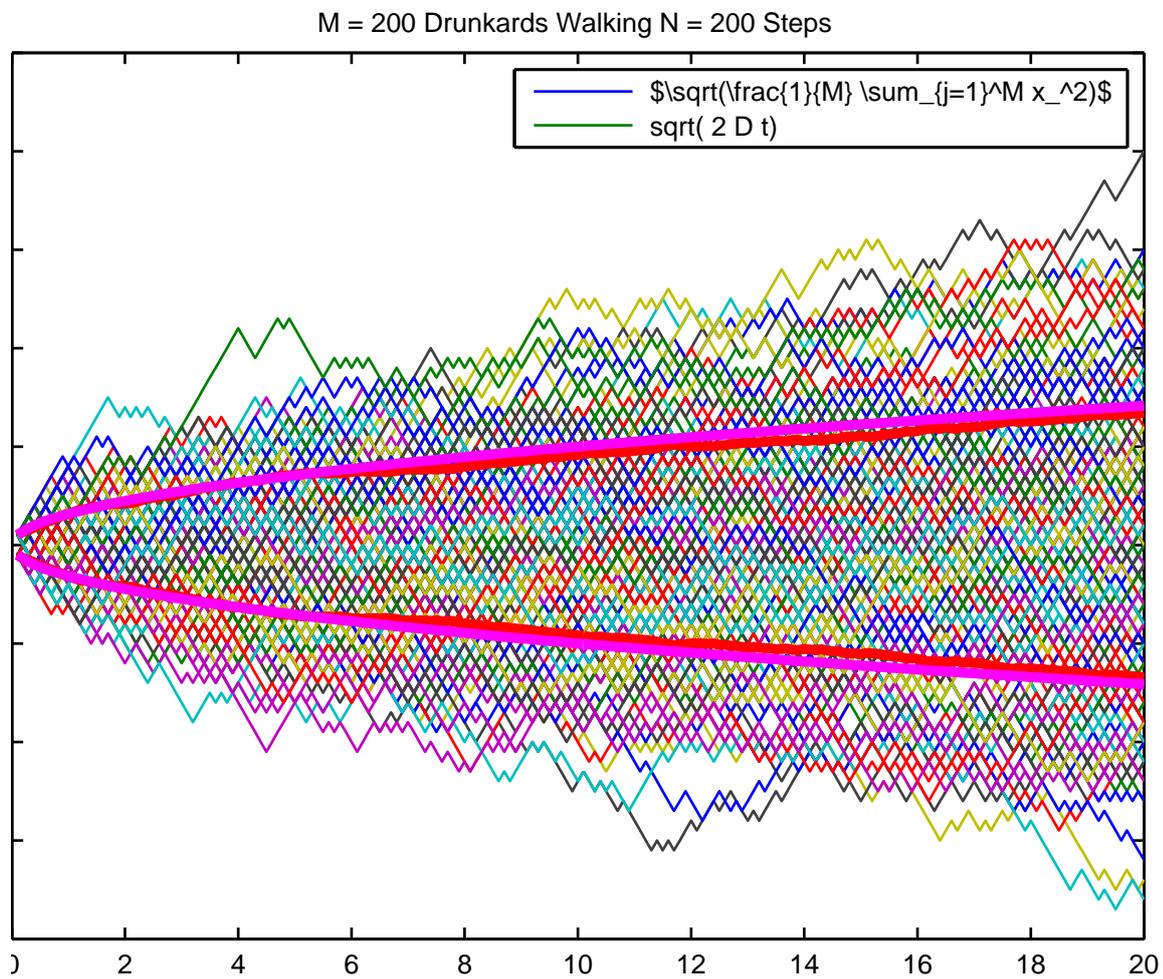


FIGURE 7. Simulation with $N = 1000$ trajectories for 200 steps and the mean absolute deviation envelopes shown in red