Lecture 10: Fourier Sine Series

(*Compiled 25 September 2018*)

In the last lecture we reduced the problem of solving the initial-boundary value problem for the heat distribution along a conducting rod to solving two ODEs, one in space and one in time. The spatial ODE and boundary conditions lead to an eigenvalue problem, which identifies a discrete set of wavenumbers $\{\lambda_n\}$ and corresponding eigenfunctions $\{X_n(x)\}$ that satisfy both homogeneous boundary conditions and the spatial ODE. Because the heat equation is linear, the general solution of the heat equation is obtained by superimposing the product of the eigensolutions and the corresponding solution of the time ODE to obtain an infinite series. All that remains is that we determine the expansion coefficients b_n for the terms of this series. These are obtained by letting $t = 0$ in the general solution and equating the resulting series to the initial value function $f(x)$. This is known as a Fourier Series. This lecture deals with the procedure to determine the Fourier coefficients b_n. Our approach is motivated by the process introduced in Linear Algebra for projecting a vector onto a set of basis vectors.

Key Concepts: Fourier Sine Series; Vector Projection; functions as infinite dimensional vectors; orthogonality; Fourier Coefficients.

10.1 Fourier Sine Series

Observe that we have a new type of eigenvalue problem in which we seek a nontrivial solution to the following boundary value problem

$$\begin{aligned} LX = -X'' = \lambda^2 X \quad &\text{or} \quad X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L). & \end{aligned} \tag{10.1}$$

Just as in the case with matrices we obtain a sequence of eigenvalues $\{\lambda_n\}$. However, because of the infinite dimensional nature of this eigenvalue the problem there are an infinite number of eigenvalues:

$$\lambda_n = \left(\frac{n\pi}{L}\right) \quad n = 1, 2, \dots \tag{10.2}$$

and corresponding eigenfunctions $\{X_n(x)\}$

$$X_n(x) = \sin \lambda_n x = \sin\left(\frac{n\pi x}{L}\right) \quad \text{or} \quad X_n(x) \in \left\{\sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{3\pi x}{L}\right), \dots\right\}. \tag{10.3}$$

In order complete the solution of the heat equation we need to determine the coefficients b_n such that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right). \tag{10.4}$$

Observations

- For symmetric matrices the eigenvalues are real - for the BVP the eigenvalues $\{\lambda_n(x)\}$ are also real.
- For symmetric matrices the eigenvectors form a basis - the eigenfunctions $\{X_n(x)\}$ are linearly independent.

10.1.1 *Euler's Column: The Buckling Load for a Beam - perhaps the oldest eigenvalue problem*

Eigenvalue problems also arise independently without necessarily coming from a PDE problem. Consider a beam that is subjected to an axial load P applied to its endpoints. Our experience tells us that as we increase P a critical load P_c is reached at which the beam starts to buckle.

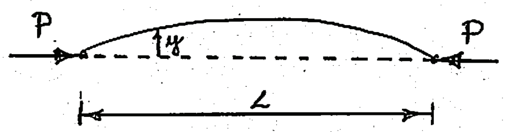


FIGURE 1. Buckling of Euler's column

The Bernoulli-Euler Law: A beam constructed from a material that is known to deform in such a way that the curvature κ is proportional to the bending moment $\kappa \alpha M$.
In particular,

$$\kappa = \frac{y''(x)}{[1+(y')^2]} = cM = \frac{M}{EI}$$

where E = Young's modulus and I = the moment of inertia of the beam. If the deflection of the beam is small $(y')^2 << |y'| << 1$ then we can make the approximation

$$\boxed{y'' = \frac{M}{EI}} \quad \text{The Bernoulli-Euler Law}$$

When subject to an axial load P as shown in figure 10.1.1, the bending moment on the buckling beam is given by

$$\begin{aligned} M(x) &= -Py(x) \\ y'' &= -\frac{M}{EI} = -\frac{P}{EI}y = -k^2 y \quad k^2 = \frac{P}{EI} \end{aligned}$$

Thus determining the magnitude of $P = EIk^2$ for which the beam will first buckle is reduced to solving the following eigenvalue problem:

$$\left.\begin{aligned} y'' + k^2 y = 0 \\ y(0) = 0 = y(L) \end{aligned}\right\} \quad \text{Eigenvalue Problem}$$

$$\begin{aligned} &y(x) = A\cos kx + B\sin kx \\ &y(0) = A = 0 \quad y(L) = B\sin kL = 0 \Rightarrow k_n = \frac{n\pi}{L} \quad n = 1, 2, \ldots. \end{aligned}$$

We have eigenvalues $k_n = \dfrac{n\pi}{L}$ and eigenfunctions $y_n(x) = \sin\left(\dfrac{n\pi x}{L}\right)$. But

$$k_n^2 = \frac{P_n}{EI} = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \ldots.$$

Therefore the smallest buckling force $P_c = P_1 = \dfrac{EI\pi^2}{L^2}$ is known as the critical Euler load and the Euler Buckling Mode is $\sin\left(\dfrac{\pi x}{L}\right)$.

10.1.2 *Finding the Fourier Coefficients*

How do we find the b_n in the sine series expansion (10.4) of $f(x)$?

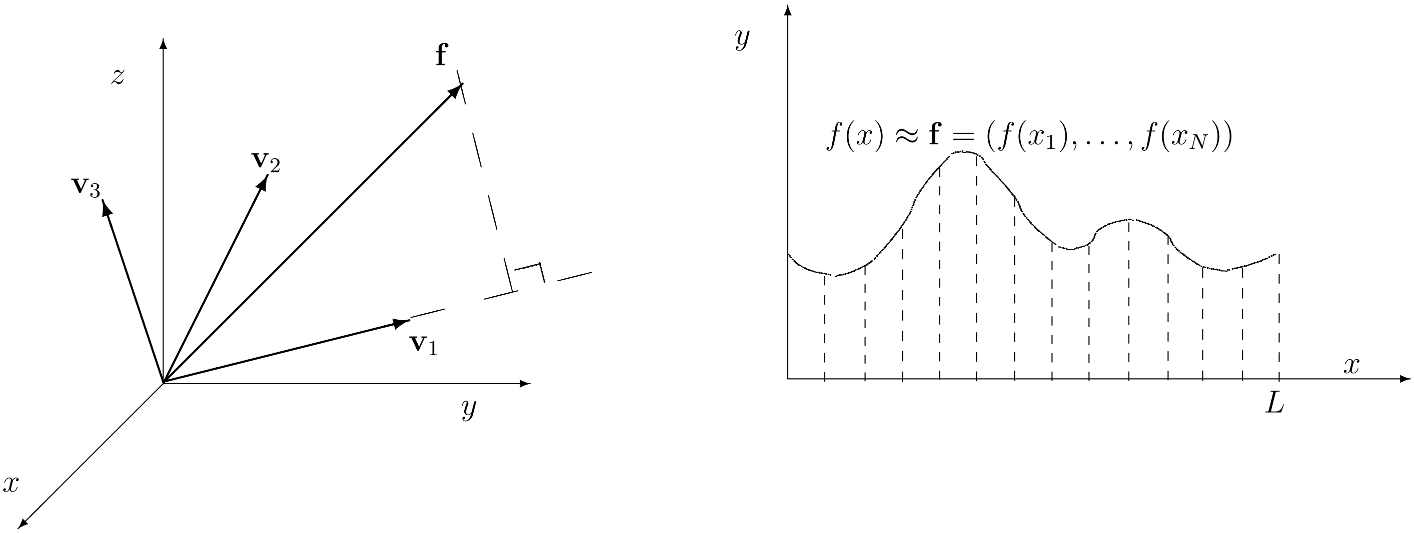


FIGURE 2. Left side: Expand $\mathbf{f}$ in terms of the basis vectors$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$; Right side: Approximation of a function $f(x)$ by a vector $\mathbf{f}$ of sample points with values $\{f(x_k\}$

Decomposition of vectors into components - projection: How do we expand a vector $\mathbf{f}$ in terms of linearly independent vectors $\mathbf{v}_k$?

$$\begin{aligned} &\text{Assume } \mathbf{f} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 \\ &\mathbf{f}\cdot\mathbf{v}_k = \alpha_1\mathbf{v}_1\cdot\mathbf{v}_k + \alpha_2\mathbf{v}_2\cdot\mathbf{v}_k + \alpha_3\mathbf{v}_3\cdot\mathbf{v}_k \\ &\begin{bmatrix} \mathbf{v}_1\cdot\mathbf{v}_1 & \mathbf{v}_1\cdot\mathbf{v}_2 & \mathbf{v}_1\cdot\mathbf{v}_3 \\ \mathbf{v}_1\cdot\mathbf{v}_2 & \mathbf{v}_2\cdot\mathbf{v}_2 & \mathbf{v}_2\cdot\mathbf{v}_3 \\ \mathbf{v}_1\cdot\mathbf{v}_3 & \mathbf{v}_2\cdot\mathbf{v}_3 & \mathbf{v}_3\cdot\mathbf{v}_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}\cdot\mathbf{v}_1 \\ \mathbf{f}\cdot\mathbf{v}_2 \\ \mathbf{f}\cdot\mathbf{v}_3 \end{bmatrix} \end{aligned} \tag{10.5}$$

If $\mathbf{v}_k \perp \mathbf{v}_\ell$, $k \neq \ell$ i.e. the $\mathbf{v}_k$ are orthogonal

$$\alpha_k = \frac{\mathbf{f}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k} \tag{10.6}$$

Functions as infinite dimensional vectors and projection: But functions are just infinite dimensional vectors:

$$\begin{aligned} \mathbf{f} &\simeq [f_1, f_2, \ldots, f_N] \\ \mathbf{g} &\simeq [g_1, g_2, \ldots, g_N] \\ \mathbf{f}\cdot\mathbf{g} &= f_1g_1 + f_2g_2 + \cdots + f_Ng_N \qquad \Delta x = \frac{L}{N} \\ &= \sum_{k=1}^{N} f(x_k)g(x_k). \end{aligned} \tag{10.7}$$

The analogue of the dot product for functions is given by the so-called *inner product*:

$$\langle f, g\rangle := \int_0^L f(x)g(x)\,dx \simeq \sum_{k=1}^{N} f(x_k)g(x_k)\Delta x. \tag{10.8}$$

Back to finding b_n:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{10.9}$$

$$\langle f, \sin\left(\frac{k\pi x}{L}\right)\rangle = \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right)\, dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right)\, dx.$$

Recall $\sin(A)\sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$. Therefore

$$\begin{aligned}
I_{nk} &= \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right)\, dx \\
&= \frac{1}{2}\int_0^L \cos(n-k)\frac{\pi x}{L} - \cos(n+k)\frac{\pi x}{L}\, dx \qquad n \neq k \\
&= \frac{1}{2}\left[\frac{\sin(n-k)\pi x/L}{(n-k)\pi/L} - \frac{\sin(n+k)\pi x/L}{(n+k)\pi/L}\right]_0^L \\
&= 0 \qquad (10.10) \\
I_{nn} &= \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\, dx = \frac{1}{2}\int_0^L 1 - \cos\left(\frac{2n\pi x}{L}\right)\, dx \\
&= L/2
\end{aligned}$$

Therefore the Fourier Coefficients $\{b_n\}$ are given by:

$$b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)\, dx. \tag{10.11}$$

Example 10.1

$$f(x) = \begin{cases} 2x & 0 < x < \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x < 1 \end{cases} \qquad L = 1$$

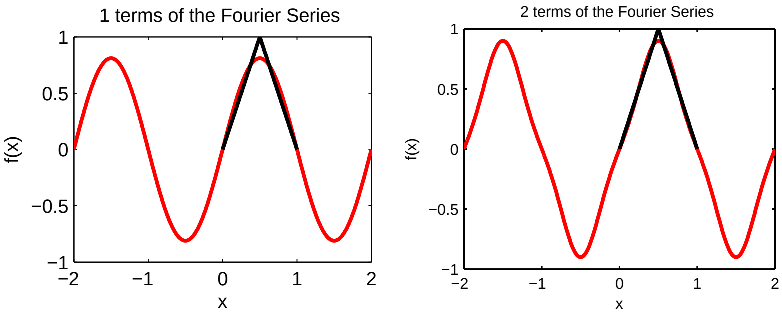
$$b_n = 2\left\{\int_0^{\frac{1}{2}} 2x\sin(n\pi x)\, dx + \int_{\frac{1}{2}}^1 2(1-x)\sin(n\pi x)\, dx\right\}$$

$$= 8\frac{\sin(n\pi/2)}{n^2\pi^2}$$

n	=	1	2	3	4	5
$\sin\left(\frac{n\pi}{2}\right)$		1	0	−1	0	1

$$\textit{Therefore } u(x,t) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left[(2k+1)\pi x\right]\mathrm{e}^{-(2k+1)^2\pi^2 t}. \tag{10.12}$$

- *Observe as $t \to \infty$ $u(x,t) \to 0$ (all the heat leaks out).*
- $u(x,0) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left[(2k+1)\pi x\right]$.
- $\frac{\pi^2}{8} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$ *by letting $x = \frac{1}{2} \Rightarrow f(x) = 1$.*



Example 10.2

$$f(x) = x \qquad 0 < x < 1 \quad L = 1$$

$$b_n = 2\int_0^1 x\sin(n\pi x)\,dx = -2\frac{\cos(n\pi)}{n\pi} = 2\frac{(-1)^{n+1}}{n\pi}$$

$$\textit{Therefore } u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)\mathrm{e}^{-(n\pi)^2 t}. \tag{10.13}$$

- *As* $t \to \infty$ $u(x,t) \to 0$.
- $u(x,0) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$.

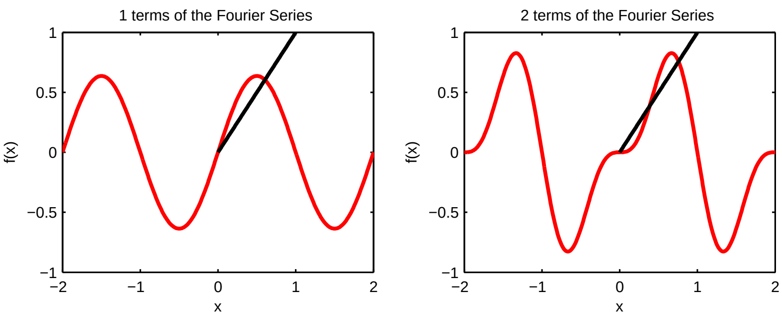
$$u\left(\frac{1}{2},0\right) = \frac{1}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi/2)$$

- $$= \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots.$$

(10.14)

k	n	$\sin\left(\frac{n\pi}{2}\right)$
0	1	1
	2	0
1	3	−1
	4	0
2	5	1



MATLAB Code:

```
%fourier sine example
clear;clf;dx=0.001;dt=0.001;
x=-2:dx:2;xr=0:dx:1;nterms=10;ntime=100;
for nt=1:ntime
    t = (nt-1)*dt;
    for n=1:nterms
        K=1:1:n;
        u(:,n+1)=2*(sin(pi*K'*x)'*((-1).^(K+1).*exp(-pi^2*K.^2*t)./K)')/pi;
        plot(x',u(:,n+1),'r-',xr',xr','k-','linewidth',2);ax=axis;ax=[0 1 0 1.2];axis(ax);
        tit=[num2str(n+1),' terms of the Fourier Series '];title(tit);xlabel('x');ylabel('u(x,t), f(x)=x');pause(.01)
    end
    if mod(nt,5)==0,pause(.02);end
end
```