

# Lecture 12: Heat equation on a circular ring - full Fourier Series

(Compiled 19 December 2017)

In this lecture we use separation of variables to solve the heat equation subject on a thin circular ring with *periodic boundary conditions*. In this case we reduce the problem to expanding the initial condition function  $f(x)$  in an infinite series of both cosine and sine functions, which we refer to as the Full Range Fourier Series.

**Key Concepts:** Heat Equation; Periodic Boundary Conditions; separation of variables; Full Fourier Series.

## 12.1 Heat equation on a circular Ring - Full Fourier Series

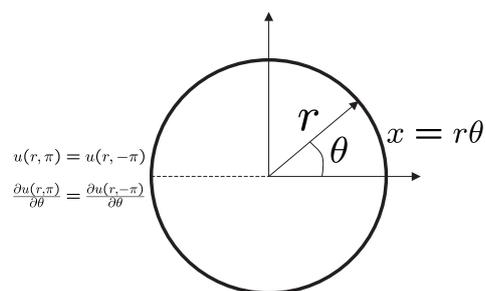


FIGURE 1. Consider a thin conducting ring with thermal conductivity  $\alpha^2$  that has a given initial temperature distribution

**Physical Interpretation:** Consider a thin circular wire in which there is no radial temperature dependence, i.e.,  $u(r, \theta) = u(\theta)$  so that  $\frac{\partial u}{\partial r} = 0$ . In this case the polar Laplacian reduces to

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{\partial^2 u}{\partial (r\theta)^2} \end{aligned} \tag{12.1}$$

and if we let  $x = r\theta$  then  $\frac{\partial^2 u}{\partial (r\theta)^2} = u_{xx}$ . In this case the heat distribution in the ring is determined by the following initial value problem with periodic boundary conditions

$$u_t = \alpha^2 u_{xx} \quad (12.2)$$

$$\begin{aligned} \text{BC:} \quad & \left. \begin{aligned} u(-L, t) &= u(L, t) \\ \frac{\partial u}{\partial x}(-L, t) &= \frac{\partial u}{\partial x}(L, t) \end{aligned} \right\} \text{Periodic BC} \\ \text{IC:} \quad & u(x, 0) = f(x) \end{aligned}$$

Assume  $u(x, t) = X(x)T(t)$ .

As before:  $\frac{X''(x)}{X(x)} = \frac{\dot{T}(t)}{\alpha^2 T(t)} = -\lambda^2$ .

IVP:  $\frac{\dot{T}(t)}{\alpha^2 T(t)} = -\lambda^2 \Rightarrow T(t) = ce^{-\lambda^2 t}$ .

$r = \frac{2L}{2\pi} = \frac{L}{\pi} = \text{Constant}$ .

BVP:  $\left. \begin{aligned} X'' + \lambda^2 X &= 0 \\ X(-L) &= X(L) \\ X'(-L) &= X'(L) \end{aligned} \right\} \begin{array}{l} \text{Eigenvalue Problem} \\ \text{look for } \lambda \text{ such that} \\ \text{nontrivial } x \text{ can be found.} \end{array}$

$$X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

$$X(-L) = A \cos(\lambda L) - B \sin(\lambda L) = A \cos(\lambda L) + B \sin(\lambda L) = X(L)$$

$$\text{therefore } 2B \sin(\lambda L) = 0$$

$$X'(x) = -A\lambda \sin \lambda x + B\lambda \cos(\lambda x) \quad (12.3)$$

$$X'(-L) = +A\lambda \sin(\lambda L) + B\lambda \cos(\lambda L) = -A\lambda \sin(\lambda L) + B\lambda \cos(\lambda L) = X'(L)$$

$$\text{therefore } 2A\lambda \sin(\lambda L) = 0$$

Therefore  $\lambda_n L = (n\pi) \quad n = 0, 1, \dots$

Solutions to (12.2) that satisfy the BC are thus of the form

$$u_n(x, t) = e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\}. \quad (12.4)$$

Superposition of all these solutions yields the general solution

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\} e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 t}. \quad (12.5)$$

In order to match the IC we have

$$f(x) = u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right). \quad (12.6)$$

As before we obtain expressions for the  $A_n$  and  $B_n$  by projecting  $f(x)$  onto  $\sin\left(\frac{n\pi x}{L}\right)$  and  $\cos\left(\frac{n\pi x}{L}\right)$ .

$$\int_{-L}^L f(x) \begin{Bmatrix} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{Bmatrix} dx = A_0 \int_{-L}^L \begin{Bmatrix} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{Bmatrix} dx \quad (12.7)$$

$$+ \sum_{n=1}^{\infty} A_n \int_{-L}^L \cos\left(\frac{n\pi x}{L}\right) \begin{Bmatrix} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{Bmatrix} dx$$

$$+ \sum_{n=1}^{\infty} B_n \int_{-L}^L \sin\left(\frac{n\pi x}{L}\right) \begin{Bmatrix} \sin\left(\frac{m\pi x}{L}\right) \\ \cos\left(\frac{m\pi x}{L}\right) \end{Bmatrix} dx.$$

As in the previous example we use the orthogonality relations:

$$\int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{mn}$$

$$\int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{mn} \quad m \text{ and } n \neq 0 \quad (12.8)$$

$$= 2L \quad m = n = 0$$

$$\int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0 \quad \forall m, n.$$

Plugging these orthogonality conditions into (12.7) we obtain

$$\left. \begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx = \text{average value of } f(x) \text{ on } [-L, L] \\ A_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \text{ and } B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned} \right\} \quad (12.9)$$

*Observations:*

(1) (12.6) and (12.9) represent the full Fourier Series Expansion for  $f(x)$  on the interval  $[-L, L]$ .

(2) By defining  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 2A_0 \\ A_n \end{cases}$  and  $b_n = B_n$  the Fourier Series (12.6) is often written in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right). \quad (12.10)$$