Lecture 15: Convergence of Fourier Series

(*Compiled 3 March 2014*)

In this lecture we state the fundamental convergence theorem for Fourier Series, which assumes that the function $f(x)$ is piecewise continuous. At points of discontinuity of $f(x)$ the Fourier Approximation $S_N(x)$ takes on the average value $\frac{1}{2}\left[f(x+)+f(x-)\right]$ and exhibits the so-called Gibbs Phenomenon in which the convergence is *pointwise but not uniform*. We explore the Gibbs phenomenon for a simple step function.

Key Concepts: Convergence of Fourier Series, Piecewise continuous Functions, Gibbs Phenomenon.

15.1 Convergence of Fourier Series

- What conditions do we need to impose on f to ensure that the Fourier Series converges to f.
- We consider piecewise continuous functions:

Theorem 1 *Let f and f' be piecewise continuous functions on $[-L, L]$ and let f be periodic with period $2L$, then f has a Fourier Series*

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = S(x) \\ \text{where} & \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \text{ and } b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned} \tag{15.1}$$

The Fourier Series converges to $f(x)$ at all points at which f is continuous and to $\frac{1}{2}\left[f(x+)+f(x-)\right]$ at all points at which f is discontinuous.

- Thus a Fourier Series converges to the average value of the left and right limits at a point of discontinuity of the function $f(x)$.

15.1.1 *Illustration of the Gibbs Phenomenon - nonuniform convergence*

- Near points of discontinuity truncated Fourier Series exhibit oscillations - overshoot.

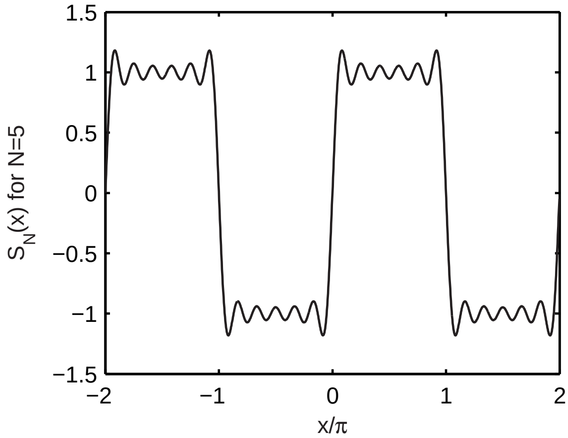

FIGURE 1. Fourier Series for a step function

Example 15.1 *Consider the half-range sine series expansion of*

$$f(x) = 1 \qquad \textit{on } [0, \pi]. \tag{15.2}$$

$$\begin{aligned}
f(x) = 1 &= \sum_{n=1}^{\infty} b_n \sin(nx) \\
\textit{where } b_n &= \tfrac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx \qquad = \tfrac{2}{\pi}\left[-\tfrac{\cos nx}{n}\right]_0^{\pi} = \tfrac{2}{\pi n}\left[1-(-1)^n\right] \\
&= \begin{cases} 4/\pi n & n \quad odd \\ 0 & n \quad even \end{cases} \\
\textit{Therefore } f(x) &= \tfrac{4}{\pi}\sum_{\substack{n=1 \\ n\ odd}}^{\infty} \tfrac{\sin(nx)}{n} \qquad = \tfrac{4}{\pi}\sum_{m=0}^{\infty}\tfrac{\sin(2m+1)x}{(2m+1)}.
\end{aligned} \tag{15.3}$$

Note:

(1) $f(\pi/2) = 1 = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin\left[(2m+1)\pi/2\right]}{(2m+1)} = \frac{4}{\pi}\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots\right\}$. Therefore $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

(2) Recall the complex Fourier Series example for the function

$$f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 < x < \pi \end{cases} \tag{15.4}$$

which turns out to be equivalent to the odd extension of the above function represented by the half-range sine expansion, which we can see from the following calculation

$$
\begin{aligned}
f(x) &= \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \tfrac{2}{\pi i n} \mathrm{e}^{inx} = \tfrac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \tfrac{\mathrm{e}^{inx}-\mathrm{e}^{-inx}}{2in} \\
&= \tfrac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \tfrac{\sin(nx)}{n}.
\end{aligned}
\tag{15.5}
$$

15.1.2 *Now consider the explicit summation of the first N terms*

$$S_N(x) = \frac{4}{\pi} \sum_{m=0}^{N} \frac{\sin(2m+1)x}{(2m+1)} = \frac{4}{\pi} Im \left\{ \sum_{m=0}^{N} \frac{\mathrm{e}^{i(2m+1)x}}{(2m+1)} \right\} \tag{15.6}$$

$$S'_N(x) = \frac{4}{\pi} Im \left\{ \sum_{m=0}^{N} i\mathrm{e}^{i(2m+1)x} \right\} \tag{15.7}$$

$$= \frac{4}{\pi} Im \left\{ i\mathrm{e}^{ix} \sum_{m=0}^{N} \left(\mathrm{e}^{i2x}\right)^m \right\} \tag{15.8}$$

$$= \frac{4}{\pi} Im \left\{ i\mathrm{e}^{ix} \left(\frac{1 + \mathrm{e}^{i2x} + \cdots + \left(\mathrm{e}^{i2x}\right)^N}{1 - \mathrm{e}^{i2x}} \right) (1 - \mathrm{e}^{i2x}) \right\} \tag{15.9}$$

$$= \frac{4}{\pi} Im \left\{ i\mathrm{e}^{ix} \left(\frac{1 - \mathrm{e}^{i2(N+1)x}}{1 - \mathrm{e}^{i2x}} \right) \right\} \tag{15.10}$$

$$= \frac{4}{\pi} Im \left\{ i \left(\frac{1 - \mathrm{e}^{i2(N+1)x}}{\mathrm{e}^{ix} - \mathrm{e}^{-ix}} \right) \right\} \tag{15.11}$$

$$= \frac{2}{\pi} Im \left\{ \frac{\mathrm{e}^{i2(N+1)x} - 1}{\sin x} \right\} \tag{15.12}$$

$$= \frac{2}{\pi} \frac{\sin 2(N+1)x}{\sin x}. \tag{15.13}$$

Therefore

$$\boxed{t = 2(N+1)u \qquad du = \tfrac{dt}{2(N+1)}} \;\swarrow$$

$$S_N(x) = \tfrac{2}{\pi} \int_0^x \tfrac{\sin 2(N+1)u}{\sin u}\, du \simeq \tfrac{2}{\pi} \int_0^{2(N+1)x} \tfrac{\sin t}{t}\, dt \tag{15.14}$$

Observe $S'_N(x) = \dfrac{2}{\pi} \dfrac{\sin 2(N+1)x}{\sin x} = 0$ when $2(N+1)x_N = \pi$ thus the maximum value of $S_N(x)$ occurs at

$$x_N = \frac{\pi}{2(N+1)} \tag{15.15}$$

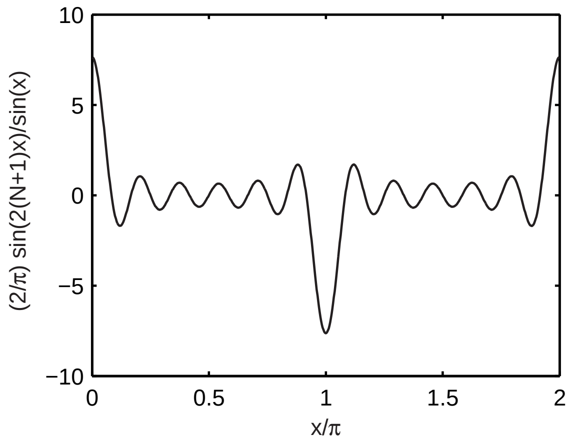

FIGURE 2. $(2/\pi)sin(2(N+1)x)/sin(x)$ for $N = 5$

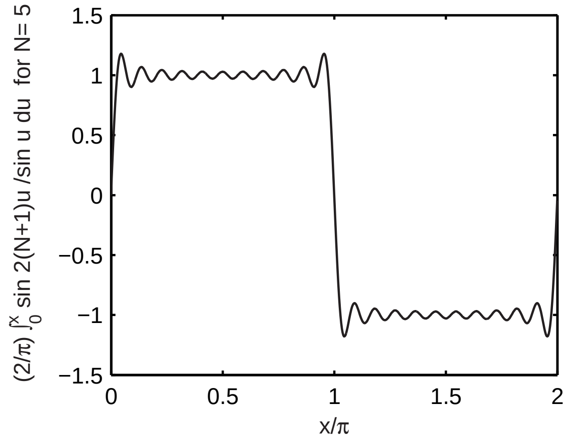

FIGURE 3. Integral of $(2/\pi)sin(2(N+1)x)/sin(x)$