Lecture 16: Bessel's Inequality, Parseval's Theorem, Energy convergence

(*Compiled 4 August 2017*)

In this lecture we consider the counterpart of Pythagoras' Theorem for functions whose square is integrable. Square integrable functions are associated with functions describing physical systems having finite energy. For a finite Fourier Series involving N terms we derive the so-called *Bessel Inequality*, in which N can be taken to infinity provided the function f is square integrable. The Bessel Inequality is shown to reduce to an equality if and only if the Fourier Series $S_n(x)$ converges to f in the energy norm. The result is known as *Parseval's Formula*, which has profound consequences for the completeness of the Fourier Basis $\{1, \cos(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L})\}$. We see that Parseval's Formula leads to a new class of sums for series of reciprocal powers of n.

Key Concepts: Convergence of Fourier Series, Bessel's Inequality, Paresval's Theorem, Plancherel theorem, Pythagoras' Theorem, Energy of a function, Convergence in Energy, completeness of the Fourier Basis.

16 Bessel's Inequality and Parseval's Theorem:

16.1 Bessel's Inequality

Definition 1 *Let $f(x)$ be a function that is square-integrable on $[-L, L]$ i.e.,*

$$\int_{-L}^{L} \left[f(x)\right]^2 dx < \infty,$$

in which case we write $f \in L_2[-L, L]$.

Consider the Fourier Series associated with $f(x)$, namely;

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = S_\infty$$

Let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Now

$$\left[f(x) - S_N(x)\right]^2 = f^2(x) - 2f(x)S_N(x) + S_N^2(x)$$

Consider the least-square error defined to be

$$\begin{aligned}
\mathcal{E}_2[f, S_N] &= \frac{1}{L}\int_{-L}^{L} \left[f(x) - S_N(x)\right]^2 dx \\
&= \frac{1}{L}\left\{\int_{-L}^{L} f^2(x)\,dx - 2\int_{-L}^{L} f(x)S_N(x)\,dx + \int_{-L}^{L} S_N^2(x)\,dx\right\} \\
&= \frac{1}{L}\left\{\langle f, f\rangle - 2\langle f, S_N\rangle + \langle S_N, S_N\rangle\right\}
\end{aligned}$$

Now

$$\begin{aligned}
\langle S_N, S_N\rangle &= \int_{-L}^{L}\left[\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]^2 dx \\
&= \frac{a_0^2}{2}L + \sum_{n=1}^{N} a_n^2 \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right)\,dx + b_n^2 \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right)\,dx \\
&= L\left[\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right]
\end{aligned}$$

In addition,

$$\begin{aligned}
\langle f, S_N\rangle &= \int_{-L}^{L} f(x)S_N(x)\,dx \\
&= \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{N} a_n \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx + b_n \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{a_0^2}{2}L + \sum_{n=1}^{N} a_n^2 L + b_n^2 L.
\end{aligned}$$

Therefore

$$\mathcal{E}_2[f, S_N] = \frac{1}{L}\int_{-L}^{L} \left[f(x) - S_N(x)\right]^2 dx = \frac{1}{L}\langle f, f\rangle - \left\{\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right\}$$

Now since $\mathcal{E}_2[f, S_N] = \int_{-L}^{L} \left[f(x) - S_N(x)\right]^2 dx \geq 0$ it follows that

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \leq \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \frac{1}{L}\langle f, f\rangle = E[f]$$

where $E[f]$ is known as the energy of the $2L$-periodic function f.

Theorem 1 *Bessel's Inequality: Let $f \in L_2[-L, L]$ then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^{L} f^2(x)\, dx$$

in particular the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$ *is convergent.*

16.2 Bessel's Inequality, Components of a Vector and Pythagoras' Theorem

16.2.1 *2D Analogue*

Consider a 2D vector f, which is decomposed into components in terms of two orthogonal unit vectors $\hat{e}_1$ and $\hat{e}_2$, i.e.

$$\tilde{f} = a_1\hat{e}_1 + a_2\hat{e}_2$$

Now

$$|f|^2 = \tilde{f} \cdot \tilde{f} = (a_1\hat{e}_1 + a_2\hat{e}_2) \cdot (a_1\hat{e}_1 + a_2\hat{e}_2)$$
$$= a_1^2 + a_2^2 \text{ since } \hat{e}_k \text{ are orthogonal unit vectors}$$

Therefore $|f|^2 = a_1^2 + a_2^2$ which is Pythagoras' Theorem.

16.2.2 *3D Analogue*

Suppose we wish to expand a 3-vector $\tilde{f}$ in terms of a set of 2 basis vectors $\{\hat{e}_1, \hat{e}_2\}$. Bessel's Inequality assumes the form

$$a_1^2 + a_2^2 \leq |f|^2$$

Since the subspace span $\{\hat{e}_1, \hat{e}_2\}$ (which represents a plane in $\mathbb{R}^3$) does not include the whole of $\mathbb{R}^3$ the vector $a_1\hat{e}_1 + a_2\hat{e}_2 \approx \tilde{f}$ represents the orthogonal projection of $\tilde{f}$ onto span $\{\hat{e}_1, \hat{e}_2\}$. If we include the third basis vector $\hat{e}_3$ in the basis, then the span $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \mathbb{R}^3$. In this case the set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are linearly independent and of full rank and thus span the complete space $\mathbb{R}^3$. $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are in this case said to form a complete set. In this case

$$\tilde{f} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

and $|\tilde{f}|^2 = a_1^2 + a_2^2 + a_3^2$ so that Bessel's Inequality assumes the form of an equality, which in this trivial case reduces to Pythagoras' Theorem. For a set of functions, that are complete, the equivalent of Pythagoras' Theorem is Parseval's Theorem.

16.3 Parseval's Theorem

Theorem 2 *(Parseval's Identity) Let $f \in L_2[-L, L]$ then the Fourier coefficients a_n and b_n satisfy Parseval's Formula*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = E[f]$$

If and only if

$$\lim_{N\to\infty}\int_{-L}^{L}\big[f(x) - S_N(x)\big]^2\,dx = 0.$$

In this case the *The Least Square Error* assumes the form

$$\begin{aligned}
\mathcal{E}_2[f, S_N] &= \frac{1}{L}\int_{-L}^{L}\big[f(x) - S_N(x)\big]^2\,dx = \frac{1}{L}\int_{-L}^{L} f^2(x)\,dx - \left(\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right) \\
&= \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right) - \left(\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right) \\
&= \sum_{n=N+1}^{\infty} a_n^2 + b_n^2 \qquad (16.1)
\end{aligned}$$

16.3.1 *Parseval's Theorem for odd functions*

Theorem 3 *(Parseval's Identity for odd functions)*

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $0 < x < L$. *Then* $\boxed{\frac{2}{L}\int_0^L \big[f(x)\big]^2\,dx = \sum_{n=1}^{\infty} b_n^2.}$

Proof:

$$\begin{aligned}
\int_0^L \big[f(x)\big]^2\,dx &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \qquad (16.2) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \cdot \delta_{mn} \cdot \frac{L}{2} = \frac{L}{2}\sum_{n=1}^{\infty} b_n^2. \qquad (16.3)
\end{aligned}$$

Example 16.1 *Recall for $x \in [0, 2]$, $f(x) = x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$. Therefore*

$$\begin{aligned}
\tfrac{2}{L}\int_0^L \big(f(x)\big)^2\,dx &= \tfrac{2}{2}\int_0^2 x^2\,dx = \left(\tfrac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\tfrac{1}{n^2} \\
\Rightarrow \quad \tfrac{x^3}{3}\Big|_0^2 &= \left(\tfrac{4}{\pi}\right)^2 \sum_{n=1}^{\infty}\tfrac{1}{n^2} \qquad (16.4) \\
\tfrac{\pi^2}{6} &= \sum_{n=1}^{\infty}\tfrac{1}{n^2}
\end{aligned}$$

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{2^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{24}$.

Also note that

$$\begin{array}{rcccc} & & \text{evens} & & \text{odds} \\ \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} & = & \sum_{m=1}^{\infty}\frac{1}{(2m)^2} & + & \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \\ & = & \frac{\pi^2}{24} & + & \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \end{array}$$

Therefore

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}. \tag{16.5}$$

For Fourier Sine Components:

$$\frac{2}{L}\int_0^L \big(f(x)\big)^2\,dx = \sum_{n=1}^{\infty} b_n^2. \tag{16.6}$$

Example 16.2 *Consider* $f(x) = x^2$, $-\pi < x < \pi$.

The Fourier Series Expansion is:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx). \tag{16.7}$$

n	1	2	3	4
$\cos\left(\frac{n\pi}{2}\right)$	0	-1	0	1

Let

$$\begin{array}{rrcl} x = \frac{\pi}{2} \quad \Rightarrow & \frac{\pi^2}{4} & = & \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi}{2}\right) \\ & -\frac{\pi^2}{12} & = & 4\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)^2} \end{array} \tag{16.8}$$

Therefore

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}. \tag{16.9}$$

By Parseval's Formula:

$$\begin{array}{rcl} \frac{2}{\pi}\int_0^{\pi} x^4\,dx & = & 2\left(\frac{\pi^2}{3}\right)^2 + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \\ \frac{2}{\pi}\left.\frac{x^5}{5}\right|_0^{\pi} & = & \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \end{array} \qquad \begin{array}{c} \frac{9-5}{45} = \frac{4}{45} = \frac{8}{90} \\ \frac{1}{90} \end{array} \tag{16.10}$$

Therefore

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\frac{1}{n^4} = \zeta(4), \tag{16.11}$$

where ζ is the Riemann Zeta Function defined by:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s},\ \ s = \sigma + (\mathrm{i})\tau,\ \ \sigma = \mathrm{Re\{s\}} > 1 \tag{16.12}$$