Lecture 18: Heat Conduction Problems with time-independent inhomogeneous BC (Cont.)

(*Compiled 3 March 2014*)

In this lecture we continue to investigate heat conduction problems with inhomogeneous boundary conditions using the methods outlined in the previous lecture.

Key Concepts: Inhomogeneous Boundary Conditions, Particular Solutions, Steady state Solutions; Separation of variables, Eigenvalues and Eigenfunctions, Method of Eigenfunction Expansions.

Reference Sections: Boyce and Di Prima Sections 10.5, 10.6, 11.2, and 11.3

18 Heat Conduction Problems with inhomogeneous boundary conditions (continued)

18.1 Heat conduction with some heat loss and inhomogeneous boundary conditions

Example 18.1 *Heat Equation with some heat loss:*

$$u_t = \alpha^2 u_{xx} - u \qquad 0 < x < L, \quad t > 0 \tag{18.1}$$

$$\text{BC: } u(0,t) = 0 \quad u(L,t) = u_1 \tag{18.2}$$

$$\text{IC: } u(x,0) = g(x). \tag{18.3}$$

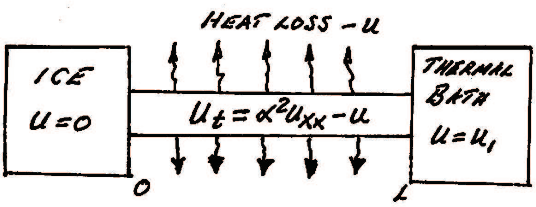
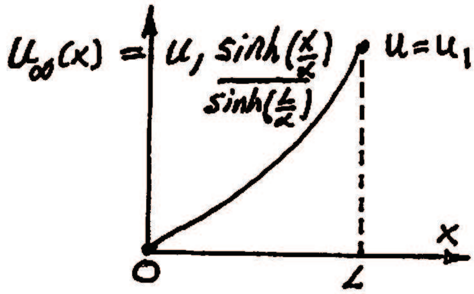


FIGURE 1. Bar subject to heat loss all along its length with inhomogeneous Mixed BC

Look for the steady state solution $u_\infty(x)$:

$$\begin{aligned} &\alpha^2 u_\infty'' - u_\infty = 0 \\ &u_\infty(x) = A\cosh\left(\frac{x}{\alpha}\right) + B\sinh\left(\frac{x}{\alpha}\right) \\ &u_\infty(0) = A = 0 \quad u_\infty(L) = B\sin h\left(\frac{L}{\alpha}\right) = u_1 \quad B = \frac{u_1}{\sin h\left(\frac{L}{\alpha}\right)}. \end{aligned} \tag{18.4}$$

Therefore



$$u_\infty(x) = u_1 \sinh\left(\frac{x}{\alpha}\right) \Big/ \sinh\left(\frac{L}{\alpha}\right). \tag{18.5}$$

Now let $u(x,t) = u_\infty(x) + v(x,t)$.

$$\begin{array}{lcl} u_t = \alpha^2 u_{xx} - u & \Rightarrow & v_t = \alpha^2 v_{xx} - v \\ u(0,t) = 0 & \Rightarrow & 0 \ = u_\infty(0) + v(0,t) \\ u(L,t) = u_1 & \Rightarrow & u_1 = u_\infty(L) + v(L,t) = u_1 + v(L,t) \\ u(x,0) = g(x) & \Rightarrow & u_\infty(x) + v(x,0) = g(x) \end{array}$$

$$\Rightarrow \quad \left\{ \begin{array}{rcl} v_t & = & \alpha^2 v_{xx} - v \\ v(0,t) & = & 0 \\ v(L,t) & = & 0 \\ v(x,0) & = & g(x) - u_\infty(x). \end{array} \right. \tag{18.6}$$

To solve (18.6) we separate variables $v(x,t) = X(x)T(t)$. Therefore

$$\frac{\dot{T}(t)}{T(t)} = \frac{\alpha^2 X''}{X} - 1 \Rightarrow \frac{1}{\alpha^2}\left(\frac{\dot{T}(t)}{T(t)} + 1\right) = \frac{X''(x)}{X(x)} = -\lambda^2. \tag{18.7}$$

Therefore

$$\dot{T}(t) = -(\lambda^2\alpha^2 + 1)T(t) \Rightarrow T(t) = c\mathrm{e}^{-(1+\lambda^2\alpha^2)t} \tag{18.8}$$

$$X'' + \lambda^2 X = 0 \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x \tag{18.9}$$

$$\Rightarrow X(0) = 0 \Rightarrow A = 0 \quad X(L) = B\sin(\lambda L) = 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)$$
$$n = 1, 2, \ldots. \tag{18.10}$$

Therefore

$$v(x,t) = \sum_{n=1}^{\infty} b_n \mathrm{e}^{-(1+\lambda_n^2\alpha^2)t} \sin\left(\frac{\pi x}{L}\right) \tag{18.11}$$

$$v(x,0) = g(x) - u_\infty(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \Rightarrow b_n \tag{18.12}$$

$$= \frac{2}{L}\int_0^L \{g(x) - u_\infty(x)\} \sin\left(\frac{n\pi x}{L}\right)\, dx. \tag{18.13}$$

Therefore

$$u(x,t) = u_1 \sinh\left(\frac{x}{\alpha}\right) \Big/ \sin h\left(\frac{L}{\alpha}\right) + \sum_{n=1}^{\infty} b_n \mathrm{e}^{-(1+\lambda_n^2\alpha^2)t} \sin\left(\frac{n\pi x}{L}\right). \tag{18.14}$$

Remark 1 Note: The $-u$ term in the PDE is responsible for the e^{-t} factor in the solution.

18.2 Heat conduction with inhomogeneous Neumann boundary conditions

Example 18.2 *Inhomogeneous Neumann BC:*

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{18.15}$$

$$\textit{BC: } u_x(0,t) = A \quad u_x(L,t) = B \tag{18.16}$$

$$\textit{IC: } u(x,0) = g(x). \tag{18.17}$$

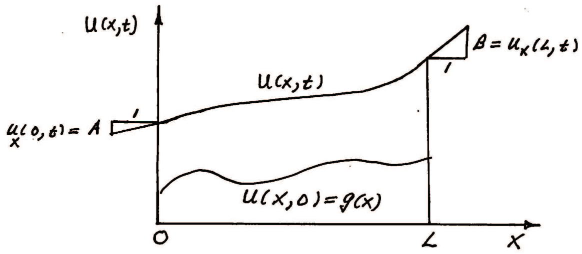


FIGURE 2. Initial, transient, and steady solutions to the heat conduction problem (18.15)-(18.17) with inhomogeneous Neumann BC

- Try for a steady solution: $u''_\infty(x) = 0$, $u_\infty(x) = \alpha x + \beta$, $u_x = \alpha$ but then we cannot match both BC unless $A = B = \alpha$. This means that if we are pumping and removing heat from the rod at different rates then the temperature does not reach a steady state.
- Instead of subtracting off a steady solution we subtract a **particular solution** which depends on x and t of the form:

$$w(x,t) = ax^2 + bx + ct \tag{18.18}$$

$$w_t = c = \alpha^2 w_{xx} = 2\alpha^2 a \Rightarrow c = 2\alpha^2 a. \tag{18.19}$$

Then

$$w(x,t) = ax^2 + bx + 2\alpha^2 at \tag{18.20}$$

solves the heat equation.

Now we determine the constants a and b so that $w(x,t)$ satisfies the inhomogeneous BC:

$$w_x = 2ax + b: \quad w_x(0,t) = b = A, \quad w_x(L,t) = 2aL + A = B. \tag{18.21}$$

Therefore $a = (B - A)/2L$. Therefore

$$w(x,t) = \frac{(B-A)}{2L}x^2 + Ax + \alpha^2 \left(\frac{B-A}{L}\right) t. \tag{18.22}$$

Now let $u(x,t) = w(x,t) + v(x,t)$

$$\begin{array}{lcll} u_t & = & w_t + v_t = \alpha^2(w_{xx} + v_{xx}) & \Rightarrow \; v_t = \alpha^2 v_{xx} \\ u_x(0,t) & = & A = w_x(0,t) + v_x(0,t) = A + v_x(0,t) & \Rightarrow \; v_x(0,t) = 0 \\ u_x(L,t) & = & B = w_x(L,t) + v_x(L,t) = B + v_x(L,t) & \Rightarrow \; v_x(L,t) = 0 \\ u(x,0) & = & g(x) = w(x,0) + v(x,0) & \Rightarrow \; v(x,0) = g(x) - w(x,0) \end{array}. \tag{18.23}$$

Equations (18.23) represent the homogeneous Neumann BVP seen previously. Therefore

$$u(x,t) = \frac{(B-A)}{2L}x^2 + Ax + \alpha^2\left(\frac{B-A}{L}\right)t + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\alpha^2\left(\frac{n\pi x}{L}\right)t} \tag{18.24}$$

where

$$a_n = \frac{2}{L}\int_0^L \left\{g(x) - \left[\frac{(B-A)}{2L}x^2 + Ax\right]\right\} \cos\left(\frac{n\pi x}{L}\right)\, dx. \tag{18.25}$$