Lecture 24: Laplace's Equation

(*Compiled 2 August 2022*)

In this lecture we start our study of Laplace's equation, which represents the steady state of a field that depends on two or more independent variables, which are typically spatial. We demonstrate the decomposition of the inhomogeneous Dirichlet Boundary value problem for the Laplacian on a rectangular domain into a sequence of four boundary value problems each having only one boundary segment that has inhomogeneous boundary conditions and the remainder of the boundary is subject to homogeneous boundary conditions. These latter problems can then be solved by separation of variables.

Key Concepts: Laplace's equation; Steady State boundary value problems in two or more dimensions; Linearity; Decomposition of a complex boundary value problem into subproblems

Reference Section: Boyce and Di Prima Section 10.8

24 Laplace's Equation

24.1 Summary of the equations we have studied thus far

In this course we have studied the solution of the second order linear PDE.

$$\begin{array}{lllllll}
\dfrac{\partial u}{\partial t} & = & \alpha^2\Delta u & \text{Heat equation:} & \text{Parabolic} & T=\alpha^2X^2 & \text{Dispersion Relation} \quad \sigma=-\alpha^2k^2 \\[2ex]
\dfrac{\partial^2 u}{\partial t^2} & = & c^2\Delta u & \text{Wave equation:} & \text{Hyperbolic} & T^2-c^2X^2=A & \text{Dispersion Relation} \quad \sigma=\pm ick \\[2ex]
\Delta u & = & 0 & \text{Laplace's equation:} & \text{Elliptic} & X^2+Y^2=A & \text{Dispersion Relation} \quad \sigma=\pm k
\end{array} \tag{24.1}$$

Important:

(1) These equations are second order because they have at most 2nd partial derivatives.

(2) These equations are all linear so that a linear combination of solutions is again a solution.

24.2 Steady state solutions in higher dimensions

Laplace's Equation arises as a steady state problem for the Heat or Wave Equations that do not vary with time so that $\dfrac{\partial u}{\partial t}=0=\dfrac{\partial^2 u}{\partial t^2}$.

2D:

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \tag{24.2}$$

3D:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{24.3}$$

- No initial conditions required.
- Only boundary conditions.

The Laplacian in Polar Coordinates: $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0.$

24.3 Laplace's Equation in two dimensions

Physical problems in which Laplace's equation arises

- 2D Steady-State Heat Conduction,
- Static Deflection of a Membrane,
- Electrostatic Potential.

$$u_t = \alpha^2(u_{xx} + u_{yy}) \longrightarrow u(x, y, t) \text{ inside a domain } D. \tag{24.4}$$

- Steady-State Solution satisfies:

$$\Delta u = u_{xx} + u_{yy} = 0 \quad (x, y) \in D \tag{24.5}$$

$$\text{BC:} \quad u \text{ prescribed on } \partial D. \tag{24.6}$$

- We consider domains D that are rectangular, circular, pizza slices.

24.3.1 *Rectangular Domains*

Consider solving the Laplace's equation on a rectangular domain (see figure 4) subject to inhomogeneous Dirichlet Boundary Conditions

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{24.7}$$

$$\text{BC:} \quad u(x, 0) = f_1(x), \quad u(a, y) = g_2(y), \quad u(x, b) = f_2(x), \quad u(0, y) = g_1(y) \tag{24.8}$$

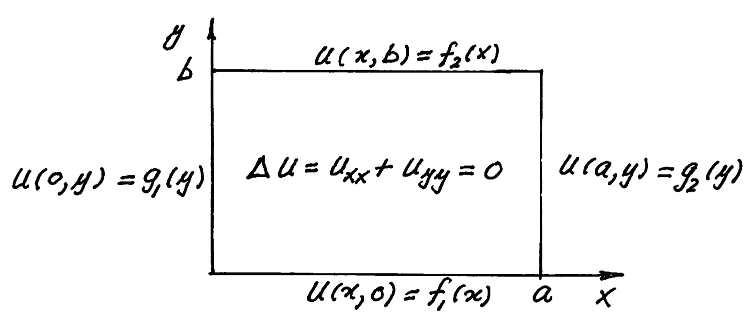

Figure 1. Inhomogeneous Dirichlet Boundary conditions on a rectangular domain as prescribed in (24.8)

Idea for solution - divide and conquer

- We want to use separation of variables so we need homogeneous boundary conditions.
- Since the equation is linear we can break the problem into simpler problems which do have sufficient homogeneous BC and use superposition to obtain the solution to (24.8).

Pictorially:

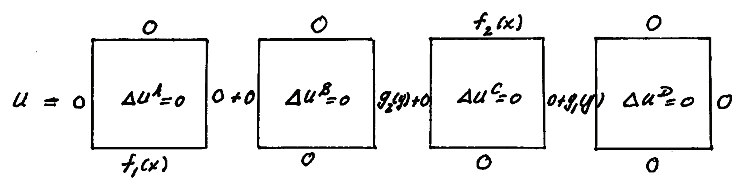

FIGURE 2. Decomposition of the inhomogeneous Dirichlet Boundary value problem for the Laplacian on a rectangular domain as prescribed in (24.8) into a sequence of four boundary value problems each having only one boundary segment that has inhomogeneous boundary conditions and the remainder of the boundary is subject to homogeneous boundary conditions

24.4 Solution to Problem (1A) by Separation of Variables

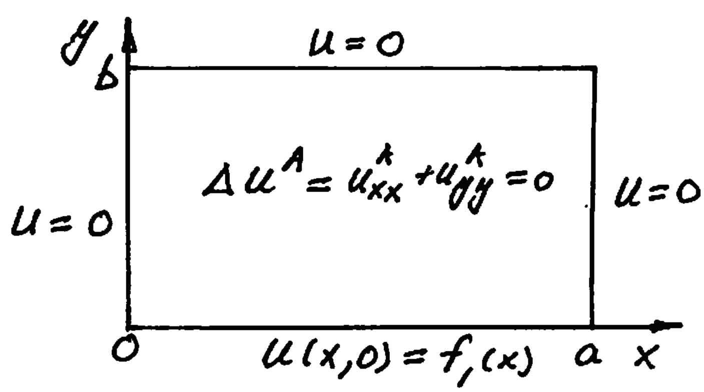

FIGURE 3. Boundary value problem for sub-solution $u^A(x, y)$

(1A)

$$u_{xx} + u_{yy} = 0 \tag{24.9}$$

$$u(0, y) = 0 = u(a, y) = u(x, b); \quad u(x, 0) = f_1(x). \tag{24.10}$$

Let

$$u(x, y) = X(x)Y(y). \tag{24.11}$$

$$X''(x)Y(y) + X(x)Y''(y) = 0 \tag{24.12}$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{ const } = \pm\lambda^2 \tag{24.13}$$

An aside about the choice of sign in front of λ^2

$-\lambda^2$**:**

$$\begin{array}{rclcl} X'' + \lambda^2 X & = & 0 & \Rightarrow X & = & A\cos(\lambda x) + B\sin(\lambda x) \text{ can match } X(0) = 0 = X(a) \\ Y'' - \lambda^2 Y & = & 0 & \Rightarrow Y & = & C\cosh(\lambda y) + D\sinh(\lambda y) \text{ can only match } Y(0) = 0 \text{ or } Y(b) = 0 \end{array} .$$

- Because sin and cos have an ∞ # of real roots $-\lambda^2$ is the appropriate choice for BC's that appear in **Problems (A) and (C)**.

$+\lambda^2$**:**

$$\begin{array}{rclcl} X'' - \lambda^2 X & = & 0 & \Rightarrow X & = & A\cosh(\lambda x) + B\sinh(\lambda x) \text{ can only match } X(0) = 0 \text{ or } X(a) = 0 \\ Y'' + \lambda^2 Y & = & 0 & \Rightarrow Y & = & C\cos(\lambda y) + D\sin(\lambda y) \text{ can match } Y(0) = 0 = Y(b) \end{array} .$$

- Again because sin and cos have an ∞ # of real roots $+\lambda^2$ is the appropriate choice for BC's that appear in **Problems (B) and (D)**.

Back to Solving (1A):

$$X(0) = 0 \Rightarrow A = 0 \tag{24.14}$$

$$X(a) = B\sin(\lambda a) = 0 \Rightarrow \boxed{\begin{array}{rcl} \lambda_n & = & \frac{n\pi}{a} \quad n = 1, 2, \ldots \\ X_n(x) & = & \sin\left(\frac{n\pi x}{a}\right) \end{array} .} \tag{24.15}$$

$$u(x, b) = X(x)Y(b) = 0 \Rightarrow Y(b) = 0 \tag{24.16}$$

$$Y(b) = C\cosh(\lambda b) + D\sinh(\lambda b) = 0 \Rightarrow C = -D\tanh(\lambda b) \tag{24.17}$$

$$Y(y) = -D\tan h(\lambda b)\cosh(\lambda y) + D\sinh(\lambda y) \tag{24.18}$$

$$= D\left\{\frac{\sinh(\lambda y)\cosh(\lambda b) - \cosh(\lambda y)\sinh(\lambda b)}{\cosh(\lambda b)}\right\} \tag{24.19}$$

$$= \frac{D}{\cosh(\lambda b)}\sinh\lambda(y - b) = \bar{D}\sinh\lambda(y - b). \tag{24.20}$$

Note: We could save ourselves the time by building the BC $Y(b) = 0$ directly into the solution by letting

$$Y_n(y) = \bar{D}\sinh\lambda_n(y - b) \tag{24.21}$$

directly. To motivate this guess. Firstly we note that we need Y(y=b)=0. Since sinh(z) only has one possible root at z=0, we have to build a function of a function out of sinh() whose argument goes to zero at y=b – y-b is just such an argument.

Now the functions: $u_n(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(y - b)\right)$ $n = 1, 2, \ldots$ satisfy all the homogeneous BC of Problem (1A). In order to match the BC $u(x, 0) = f_1(x)$ we need to superimpose all these solutions.

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(y - b)\right) \tag{24.22}$$

$$f_1(x) = u(x, 0) = \sum_{n=1}^{\infty} \underbrace{\left\{-B_n \sinh\left(\frac{n\pi b}{a}\right)\right\}}_{b_n} \sin\left(\frac{n\pi x}{a}\right) \tag{24.23}$$

where

$$-B_n \sinh\left(\frac{n\pi b}{a}\right) = b_n = \frac{2}{a}\int_0^a f_1(x)\sin\left(\frac{n\pi x}{a}\right)\,dx. \tag{24.24}$$

Therefore

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi}{a}(y-b)\right)\sin\left(\frac{n\pi x}{a}\right);$$
$$\text{where } B_n = -\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f_1(x)\sin\left(\frac{n\pi x}{a}\right)\,dx \tag{24.25}$$

Specific Example Let $f_L(x) = 1 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right)$.

$$b_n = \frac{2}{an\pi}\left[1+(-1)^{n+1}\right] = -B_n\sinh\left(\frac{n\pi b}{a}\right). \tag{24.26}$$

Therefore

$$u(x,y) = \frac{1}{a}\sum_{n=1}^{\infty} -\frac{2}{n\pi}\frac{\left[1+(-1)^{n+1}\right]}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(y-b)\right). \tag{24.27}$$

24.5 Solution to Problem (1B) by Separation of Variables

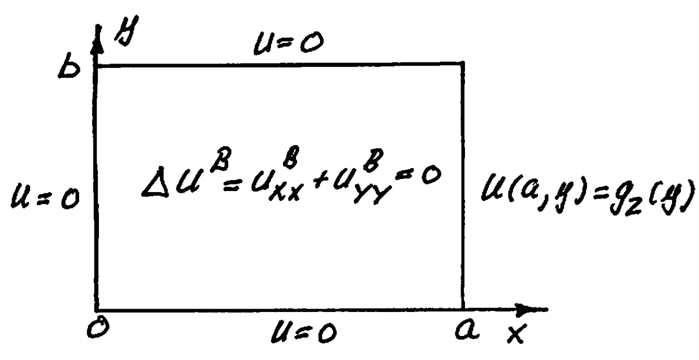

FIGURE 4. Boundary value problem for sub-solution $u^A(x,y)$

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{24.28}$$
$$0 = u(x,0) = u(x,b) = u(0,y); \quad u(b,y) = g_2(y) \tag{24.29}$$

Let

$$u(x,y) = X(x)Y(y) \tag{24.30}$$
$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \pm\lambda^2. \tag{24.31}$$

Since we have homogeneous BC at $y = 0$ and $y = b$ we want the function $Y(y)$ to behave like sines and cosines. So we choose const $= +\lambda^2$

$$\begin{array}{rclrcl} X'' - \lambda^2 X &=& 0 \qquad & X &=& c_1 \cosh \lambda x + c_2 \sinh \lambda x \\ Y'' + \lambda^2 Y &=& 0 \qquad & Y &=& A\cos(\lambda y) + B\sin(\lambda y) \end{array} \tag{24.32}$$

$$u(x,0) = X(x)Y(0) = 0 \Rightarrow Y(0) = 0 \Rightarrow Y(0) = A = 0 \tag{24.33}$$

$$u(x,b) = X(x)Y(b) = 0 \Rightarrow Y(b) = 0 \Rightarrow Y = B\sin(\lambda b) = 0, \Rightarrow \boxed{\begin{array}{rcl} \lambda_n &=& \frac{n\pi}{b} \quad n = 1, 2, \ldots \\ Y_n(y) &=& \sin\left(\frac{n\pi y}{b}\right) \end{array}.} \tag{24.34}$$

$$u(0,y) = X(0)Y(y) = 0 \Rightarrow X(0) = c_1 = 0$$

Therefore $X_n(x) = c_2 \sinh\left(\dfrac{n\pi x}{b}\right)$.
Therefore $u_n(x,y) = \sin\left(\dfrac{n\pi y}{b}\right)\sinh\left(\dfrac{n\pi x}{b}\right)$ satisfy the homogeneous BC.
Therefore $u(x,y) = \displaystyle\sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$.
Now to satisfy the inhomogeneous BC

$$g_2(y) = u(a,y) = \sum_{n=1}^{\infty} \underbrace{c_n \sinh\left(\frac{n\pi a}{b}\right)}_{b_n} \sin\left(\frac{n\pi y}{b}\right) \tag{24.35}$$

where

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b g_2(y)\sin\left(\frac{n\pi y}{b}\right)\, dy. \tag{24.36}$$

Summarizing:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right); \; c_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b g_2(y)\sin\left(\frac{n\pi y}{b}\right)\, dy. \tag{24.37}$$