Lecture 27: Wedges with cut-outs, Dirichlet and Neumann problems on Circular domains

(*Compiled 4 August 2017*)

In this lecture we continue with the solution of Laplace's equation on circular domains. In particular, a wedge-shaped domain with a circular cut-out, i.e., a pizza slice with a bite taken out of it, as well as the Dirichlet and Neumann problems on the interior of a circle. This leads to the well known *Poisson Integral Formula* and an interesting application known as *Electrical Impedance Tomography (EIT).*

Key Concepts: Laplace's equation; Circular domains; Pizza Slice-shaped regions; Dirichlet and Neumann Boundary Conditions, Poisson Integral Formula; Applications to EIT.

Reference Section: Boyce and Di Prima Section 10.8

27 Wedges with cut-outs, circles, holes, and annuli

27.1 Wedges with cut-outs

Example 27.1 *A circular wedge with a cut-out:*

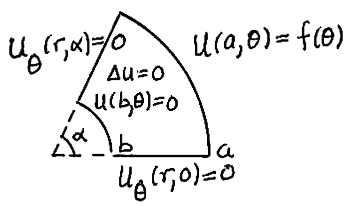


FIGURE 1. Mixed boundary conditions on a wedge shaped domain with a cut-out (27.2)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{27.1}$$

$$\begin{array}{ll} u_\theta(r,0) = 0 & u_\theta(r,\alpha) = 0 \\ u(b,\theta) = 0 & u(a,\theta) = f(\theta) \end{array} \tag{27.2}$$

Let $u(r,\theta) = R(r)\Theta(\theta)$.

$$\frac{r^2(R'' + \frac{1}{r}R)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda^2 \Rightarrow \begin{cases} r^2R'' + rR' - \lambda^2 R = 0 \\ \Theta'' + \lambda^2\Theta = 0 \end{cases} \tag{27.3}$$

Θ equation⟩

$$\left.\begin{array}{l}\Theta''+\lambda^2\Theta=0\\ \Theta'(0)=0=\Theta'(\alpha)\end{array}\right\}\begin{array}{lcl}\Theta&=&A\cos\lambda\theta+B\sin\lambda\theta\\ \Theta'(0)&=&B\lambda=0\Rightarrow B\text{ or }\lambda=0,\end{array}\tag{27.4}$$

$$\begin{array}{lcl}\Theta'&=&-A\lambda\sin\lambda\theta+B\lambda\cos\lambda\theta\\ \Theta'(\alpha)&=&-A\lambda\sin\lambda\alpha=0,\ \lambda=\frac{n\pi}{\alpha}\ n=0,1,\dots\end{array}\tag{27.5}$$

R equation⟩ $\mathbf{n=0}$: $(rR_0')'=0$ $rR_0'=B_0$ $R_0=A_0+B_0\ln r$. Note

$$u_0(b,\theta)=R_0(b)\Theta_0(\theta)=0\Rightarrow R_0(b)=A_0+B_0\ln b=0, A_0=-B_0\ln b.\tag{27.6}$$

Therefore $R_0=B_0\ln(r/b)$. Choose $B_0=1$.
$\mathbf{n\geq 1}$: $r^2R_n''+rR_n'-\lambda^2R_n=0$ $R(r)=A_nr^{\lambda_n}+B_nr^{-\lambda_n}$

$$R_n(b)=A_nb^{\lambda_n}+B_nb^{-\lambda_n}=0\Rightarrow B_n=-A_nb^{2\lambda_n}\tag{27.7}$$

$$R_n(r)=A_n[r^{\lambda_n}-b^{2\lambda_n}r^{-\lambda_n}]\quad\text{Choose }A_n=1.\tag{27.8}$$

$$u_n(r,\theta)=\left[r^{\left(\frac{n\pi}{\alpha}\right)}-b^{2\left(\frac{n\pi}{\alpha}\right)}r^{-\left(\frac{n\pi}{\alpha}\right)}\right]\cos\left(\frac{n\pi\theta}{\alpha}\right)\tag{27.9}$$

$$u_0(r,\theta)=\ln\left(\frac{r}{b}\right)\cdot 1\tag{27.10}$$

Therefore

$$u(r,\theta)=c_0\ln\left(\frac{r}{b}\right)+\sum_{n=1}^{\infty}c_n\left[r^{\left(\frac{n\pi}{\alpha}\right)}-b^{\left(\frac{2n\pi}{\alpha}\right)}r^{-\left(\frac{n\pi}{\alpha}\right)}\right]\cos\left(\frac{n\pi\theta}{\alpha}\right)\tag{27.11}$$

$$u(a,\theta)=f(\theta)=2\frac{\left(c_0\ln\left(\frac{a}{b}\right)\right)}{2}+\sum_{n=1}^{\infty}c_n\left[a^{\left(\frac{n\pi}{\alpha}\right)}-b^{\left(\frac{2n\pi}{\alpha}\right)}r^{-\left(\frac{n\pi}{\alpha}\right)}\right]\cos\left(\frac{n\pi\theta}{\alpha}\right)\tag{27.12}$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi\theta}{\alpha}\right).\tag{27.13}$$

Therefore

$$2c_0\ln(a/b)=\frac{2}{\alpha}\int_0^{\alpha}f(\theta)\,d\theta.\tag{27.14}$$

$$c_n=\frac{2}{\alpha\left[a^{\left(\frac{n\pi}{\alpha}\right)}-b^{\left(\frac{2n\pi}{\alpha}\right)}a^{-\frac{n\pi}{\alpha}}\right]}\int_0^{\alpha}f(\theta)\cos\left(\frac{n\pi\theta}{\alpha}\right)\,d\theta\tag{27.15}$$

$$c_0=\frac{1}{\alpha\ln(a/b)}\int_0^{\alpha}f(\theta)\,d\theta.\tag{27.16}$$

Observations: In the special case $f(\theta)=1$, $c_0=\dfrac{1}{\log(a/b)}$, and $c_n=0$ $n\geq 1$. By Fourier basis function orthogonality $\int_0^{\alpha}1\cdot\cos\left(\frac{n\pi\theta}{\alpha}\right)\,d\theta=0$ so that the solution reduces to:

$$u(r,\theta)=\frac{\log(r/b)}{\log(a/b)}\tag{27.17}$$

which is purely radial i.e. has no θ dependence.

27.2 Problems with a complete circle as the boundary

Example 27.2 *Dirichlet problem in the interior of a circle*

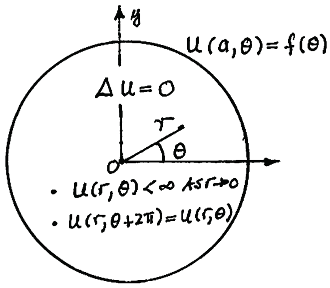


FIGURE 2. Dirichlet boundary condition prescribed on the on boundary of a circle (**??**)

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad 0 < r < a, \quad 0 < \theta < 2\pi. \tag{27.18}$$

$$\begin{array}{l} \textit{BC: } u(a,\theta) = f(\theta) \qquad u(r,\theta) < \infty \quad r \to 0 \\ \textit{Periodicity } u(\theta + 2\pi) = u(\theta) \quad \textit{periodic.} \end{array} \tag{27.19}$$

Let $u(r,\theta) = R(r)\Theta(\theta)$.

$$\frac{r^2\left(R'' + \frac{1}{r}R'\right)}{R(r)} = -\frac{\Theta''}{\Theta} = +\lambda^2 \Rightarrow \begin{array}{l} r^2R'' + rR' - \lambda^2 R = 0 \text{ Euler Eq.} \\ \Theta'' + \lambda^2\Theta = 0 \end{array} \tag{27.20}$$

$\boldsymbol{\Theta}$ equation$\rangle$ $\Theta'' + \lambda^2\Theta = 0$ $\Theta = A\cos(\lambda\theta) + B\sin(\lambda\theta)$

$$\theta(-\pi) = A\cos(\lambda\pi) - B\sin(\lambda\pi) = \Theta(\pi) = A\cos(\lambda\pi) \tag{27.21}$$

$$+B\sin(\lambda\pi) \Rightarrow 2B\sin(\lambda\pi) = 0 \tag{27.22}$$

$$\Theta'(\theta) = -A\lambda\sin(\lambda\theta) + B\lambda\cos(\lambda\theta) \tag{27.23}$$

$$\Theta'(-\pi) = A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi) = \Theta'(\pi) = -A\lambda\sin(\lambda\pi) \tag{27.24}$$

$$+B\lambda\cos(\lambda\pi) \Rightarrow 2A\lambda\sin(\lambda\pi) = 0. \tag{27.25}$$

Therefore $\lambda_n = \dfrac{n\pi}{\pi} = n$; $n = 0, 1, 2, \ldots$ $\Theta_n(\theta) = \cos(n\theta)$ and $\sin(n\theta)$.
R equation$\rangle$ $\mathbf{n = 0}$: $\lambda_0 = 0 \Rightarrow rR_0'' + R_0' = (rR_0')' = 0$ $R_0 = B_0 \ln r + A_0$.

$\mathbf{n \geq 1}$: $\lambda_\mathbf{n} = \mathbf{n}$:

$$\begin{array}{l} r^2R_n'' + rR_n' - (n)^2R_n = 0 \quad \text{Euler Eq.} \\ R_n = r^\gamma \Rightarrow \gamma(\gamma-1) + \gamma - n^2 = 0 \quad \gamma = \pm\lambda_n = \pm n \\ R_n = c_n r^{-n} + d_n r^n. \end{array} \tag{27.26}$$

Now since $u(r,\theta) < \infty$ as $r \to 0$ we must exclude solutions that blow up. Thus $B_0 = 0$ and $c_n = 0$. Therefore

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\theta) + b_n \sin(n\theta)\} r^n \tag{27.27}$$

$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] a^n \tag{27.28}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \quad a_n = \frac{a^{-n}}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta \tag{27.29}$$

$$b_n = \frac{a^{-n}}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta. \tag{27.30}$$

The Poisson's Integral Formula:

Using the geometric series it is possible to sum the solution (27.27) explicitly. We proceed as follows:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(n\theta) + b_n \sin(n\theta)\big) r^n \tag{27.31}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi} f(\phi) \big[\cos n\theta \cos n\phi + \sin(n\theta)\sin(n\phi)\big]\, d\phi \tag{27.32}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{2} \int_{-\pi}^{\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \int_{-\pi}^{\pi} f(\phi) \cos n(\theta - \phi)\, d\phi \right\} \tag{27.33}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right\} d\phi \tag{27.34}$$

Now

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\tfrac{r}{a}\right)^n \cos n(\theta-\phi) &= Re \sum_{n=1}^{\infty} z^n \\ &= Re \left(\tfrac{z}{1-z}\right) \\ &= Re \left(\tfrac{z(1-\bar{z})}{(1-z)(1-\bar{z})}\right) \\ &= \frac{\left(\frac{r}{a}\right)\cos(\theta-\phi) - \left(\frac{r}{a}\right)^2}{1 - 2\left(\frac{r}{a}\right)\cos(\theta-\phi) + \left(\frac{r}{a}\right)^2} \\ &= \frac{ar\cos(\theta-\phi) - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2} \end{aligned}$$

$$\begin{aligned} z &= \left(\tfrac{r}{a}\right) e^{i(\theta-\phi)} \\ (1-z)(1-\bar{z}) &= 1 - (z+\bar{z}) + |z|^2 \\ &= 1 - 2\left(\tfrac{r}{a}\right)\cos(\theta-\phi) + \left(\tfrac{r}{a}\right)^2 \\ z(1-\bar{z}) &= \tfrac{r}{a} e^{i(\theta-\phi)} - \left(\tfrac{r}{a}\right)^2 \end{aligned}$$

Therefore

$$u(r,\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \left\{ \frac{1}{2} + \frac{ar\cos(\theta-\phi) - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2} \right\} d\phi \tag{27.35}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \left\{ \frac{\frac{1}{2}a^2 - ar\cos(\theta-\phi) + \frac{1}{2}r^2 + ar\cos(\theta-\phi) - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2} \right\} d\phi \tag{27.36}$$

$$u(r,\theta) = \frac{1}{2\pi}(a^2 - r^2) \int_{-\pi}^{\pi} \frac{f(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\, d\phi \tag{27.37}$$

Domain exterior to a circle:

For problem exterior to a circle we require that $u(r,\theta) < \infty$ as $r \to \infty$. In this case we require that $B_0 = 0$ and that $d_n = 0$ so that $R_0 = A_0$ and $R_n = r^{-n} \cdot \left[a_n \cos n\theta + b_n \sin(n\theta)\right]$. In this case the appropriate solution to the Dirichlet problem becomes:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] r^{-n} \tag{27.38}$$

$$a_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta, b_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta. \tag{27.39}$$

Example 27.3 *Neumann problem on the interior of a circle*

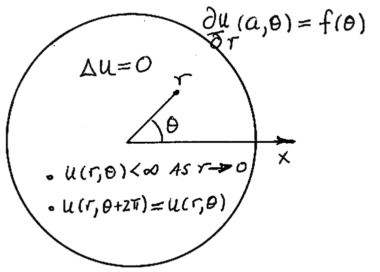


FIGURE 3. Neumann boundary condition prescribed on the on boundary of a circle (27.40)

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ &\frac{\partial u}{\partial r}(a,\theta) = f(\theta) \\ &u \quad 2\pi \text{ - periodic.} \end{aligned} \tag{27.40}$$

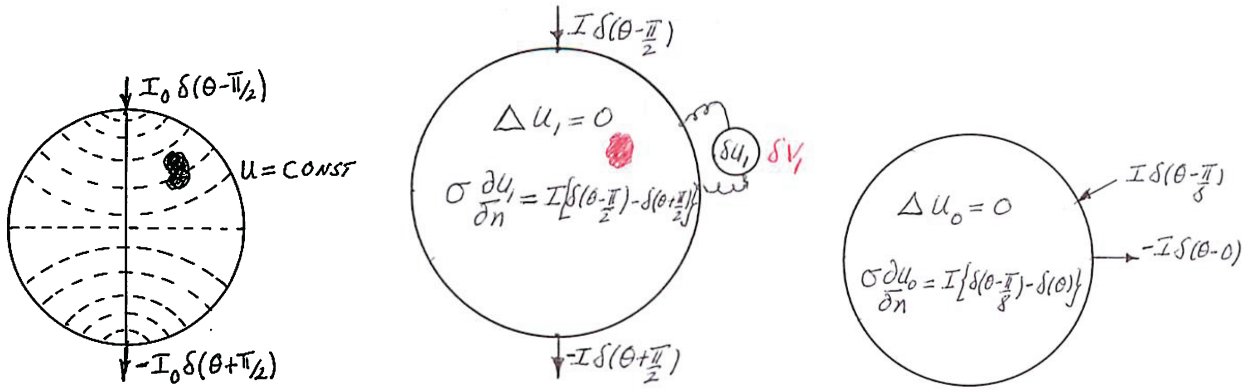
$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] \tag{27.41}$$

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = f(\theta) = \sum_{n=1}^{\infty} n r^{n-1} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]\Big|_{r=a} \tag{27.42}$$

$$= \sum_{n=1}^{\infty} n a^{n-1} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]. \tag{27.43}$$

A solution will not exist unless $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = 0$. Otherwise there is a net flux of heat across the boundary and no steady state solution will exist.

Special Case - Electrical Impedance Tomography (EIT)
Assume that $f(\theta) = I_0\delta\left(\theta - \frac{\pi}{2}\right) - I_0\delta\left(\theta + \frac{\pi}{2}\right)$.



(a) Level sets of potential field (27.61) (b) BVP for U_1 field and voltage measurements δU_1 with $\sigma = const$. Actual voltage δV_1 with a feature (red) (c) Field U_0 used in the sensitivity Thm

FIGURE 4. Sensitivity Thm: $\delta V_1 - \delta U_1 \simeq -\frac{1}{I}\int_\Omega \nabla U_1 \cdot \nabla U_0 \delta\sigma dv$ & backprojection rule: $\delta\sigma \simeq -\frac{(\delta V_1 - \delta U_1)}{\delta U_1}\sigma$

$$f(-\theta) = I_0\delta\left(-\left(\theta + \frac{\pi}{2}\right)\right) - I_0\delta\left(-\left(\theta - \frac{\pi}{2}\right)\right) \tag{27.44}$$
$$= I_0\delta\left(\theta + \frac{\pi}{2}\right) - I_0\delta\left(\theta - \frac{\pi}{2}\right) = -f(\theta) \tag{27.45}$$

Thus f is odd $\Rightarrow a_0 = a_n = 0$.

$$na^{n-1}b_n = \frac{2}{\pi}\int_0^\pi I_0\delta\left(\theta - \frac{\pi}{2}\right)\sin(n\theta)\,d\theta \tag{27.46}$$
$$b_n = \frac{2I_0}{\pi n a^{n-1}}\sin\left(\frac{n\pi}{2}\right) \tag{27.47}$$
$$u(r,\theta) = \frac{2aI_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n}\sin\left(\frac{n\pi}{2}\right)\left(\frac{r}{a}\right)^n \tag{27.48}$$

$$\text{For enrichment } \downarrow = \frac{2aI_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\frac{1}{2}\left[\frac{\cos n\left(\theta - \frac{\pi}{2}\right)}{n} - \frac{\cos n\left(\theta + \frac{\pi}{2}\right)}{n}\right] \tag{27.49}$$
$$= \frac{aI_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\frac{\cos n\left(\theta - \frac{\pi}{2}\right)}{n} - \frac{\cos n\left(\theta + \frac{\pi}{2}\right)}{n}\right] \tag{27.50}$$
$$= \frac{aI_0}{\pi}Re\left[\sum_{n=1}^{\infty}\frac{z_1^n}{n} - \sum_{n=1}^{\infty}\frac{z_2^n}{n}\right] \quad \begin{array}{rcl} z_1 & = & \left(\frac{r}{a}\right)e^{i\left(\theta - \frac{\pi}{2}\right)} \\ z_2 & = & \left(\frac{r}{a}\right)e^{i\left(\theta + \frac{\pi}{2}\right)} \end{array}. \tag{27.51}$$

Now for $|\mathbf{z}| < \mathbf{1}$:

$$\begin{array}{rcccl} \dfrac{1}{1-z} & = & 1 + z + z^2 + \cdots & = & \displaystyle\sum_{k=0}^{\infty} z^k \\ -\ln(1-z) & = & z + \dfrac{z^2}{2} + \cdots & = & \displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n} \end{array}. \tag{27.52}$$

Therefore

$$u(r,\theta) = -\frac{aI_0}{\pi} Re\left[\ln\left(\frac{1-z_1}{1-z_2}\right)\right]. \tag{27.53}$$

Now if $(1-z) = A\mathrm{e}^{i\phi}$ then

$$Re\big[\ln(1-z)\big] = Re\left[\ln\left(A\mathrm{e}^{i\phi}\right)\right] = Re[\ln A + i\phi] = \ln A. \tag{27.54}$$

Therefore

$$u(r,\theta) = -\frac{aI_0}{2\pi}\ln\left|\frac{1-z_1}{1-z_2}\right|^2. \tag{27.55}$$

Now

$$z_1 = \left(\frac{r}{a}\right)\mathrm{e}^{i\left(\theta-\frac{\pi}{2}\right)} = \rho\mathrm{e}^{i\phi_1} \tag{27.56}$$

$$\|1-z_1\|^2 = (1-z_1)\overline{(1-z_1)} = \left(1-\rho\mathrm{e}^{i\phi_1}\right)\left(1-\rho\mathrm{e}^{-i\phi_1}\right) \tag{27.57}$$

$$= 1-\rho\left(\mathrm{e}^{i\phi_1}+\mathrm{e}^{-i\phi_1}\right)+\rho^2 \tag{27.58}$$

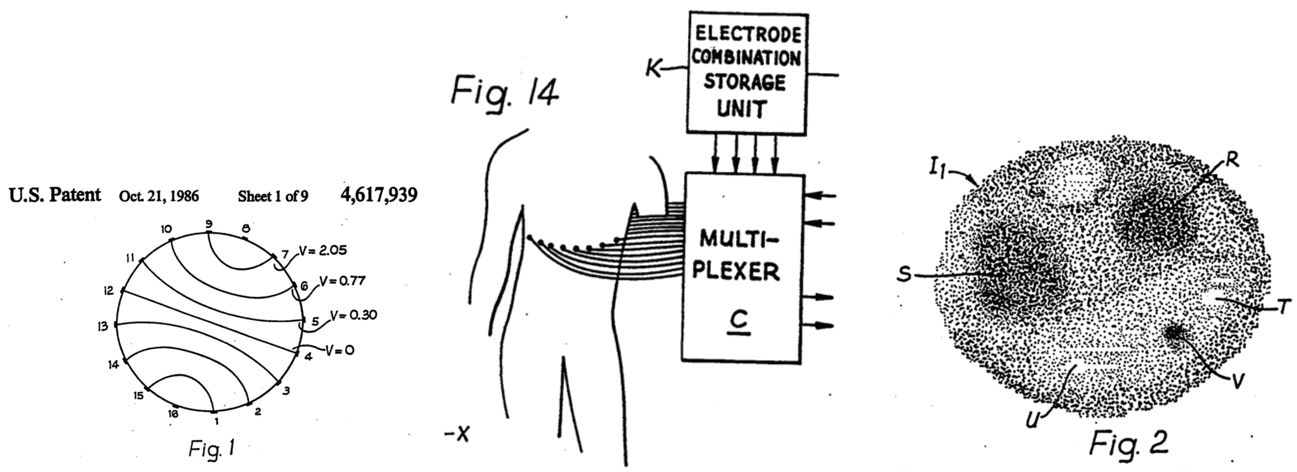
$$= 1-2\rho\cos\phi_1+\rho^2. \tag{27.59}$$

Similarly $z_2 = \left(\frac{r}{a}\right)\mathrm{e}^{i\left(\phi+\frac{\pi}{2}\right)} = \rho\mathrm{e}^{i\phi_2}$ and $|1-z_2|^2 = 1-2\rho\cos\phi_2+\rho^2$. Therefore

$$u(r,\theta) = \frac{aI_0}{2\pi}\ln\left[\frac{1-2(\frac{r}{a})\cos(\theta+\frac{\pi}{2})+(\frac{r}{a})^2}{1-2(\frac{r}{a})\cos(\theta-\frac{\pi}{2})+(\frac{r}{a})^2}\right] \tag{27.60}$$

$$u(r,\theta) = \frac{aI_0}{2\pi}\ln\left[\frac{a^2+2ar\sin\theta+r^2}{a^2-2ar\sin\theta+r^2}\right] \tag{27.61}$$

Electrical Impedance Tomography - from the US Patent



(a) Level sets of potential field used in the US Patent

(b) Medical imaging device

(c) Actual image obtained from EIT

FIGURE 5. Figures taken from a US Patent for a medical imaging device based in EIT and the level sets of the solution given in (27.61)

The Economist

Technology Quarterly: Q1 2012

In this Technology Quarterly

Monitor

We’ve got it covered

Materials: Smart paint promises to make it easier to identify and repair cracks or corrosion in bridges and other infrastructure

Mar 3rd 2012 | from the print edition

IT WOULD be much easier to locate and repair damage to bridges, wind turbines and other dumb objects if those objects could tell you what the problem was. Researchers at the University of Strathclyde, in Britain, led by Mohamed Saafi, are therefore trying to give them a voice, by devising a new sort of smart paint.

It is composed of what sounds like a bizarre mixture: fly-ash, a fine-grained waste product from coal-fired power stations; carbon nanotubes, cylindrical molecules made of elemental carbon; and two binding agents, sodium silicate and sodium hydroxide. The result is a material similar to cement, which makes a suitably tough paint. When it dries, the fly-ash acts as a coating, able to withstand the elements in exposed places. The carbon nanotubes are there to conduct electricity.

The smart bit is that the tubes' conductivity is affected by cracks in, or corrosion of, the painted surface. When put under stress, for example, the nanotubes bend and become less conductive. If inundated by chloride ions, as a result of corrosion by salt water, their conductivity increases. This makes it possible to monitor damage.

The voltage running through any part of the painted area can be measured remotely, with an array of electrodes distributed across its surface, and data for the entire structure dispatched, via a central transmitter, to a computer. Using a medical-imaging technique called electrical-impedance tomography, Dr Safi and his colleague David McGahon are devising software that can draw a conductivity map of an entire painted structure.

Nanotechnology has been used in paints before. Sometimes the goal is to bind the paint tightly to the material it has been applied to. Sometimes it is to channel water molecules efficiently, thus keeping a surface clean. Some paints incorporate tiny silver particles, which capture atmospheric pollutants. But Dr Saafi's smart paint is new in several ways.

It is cheap, so it is possible to imagine whole structures being built out of it, instead of cement. It is also versatile, theoretically able to detect a broad range of stresses and pollutants. If it performs well, there are currently 3,500 wind turbines—and counting—in Britain alone that could do with a lick of it.

from the print edition | Technology Quarterly