# Lecture 30: Sturm-Liouville Problems involving the Cauchy-Euler Equation - Applications

(*Compiled 4 August 2017*)

In this lecture we look at eigenvalue problems involving equidimensional or Cauchy-Euler differential operators.

**Key Concepts:** Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Cauchy-Euler Equations; Equidimensional equations.

**Reference Section: Boyce and Di Prima Section 11.1 and 11.2**

## 30 Variable coefficient BVP - eigenfunctions involving solutions to the Euler Equation:

**Example 30.1** *Eigenfunctions involving solutions to an Euler Equation:*

$$\begin{array}{ll} (x^2\phi')' + \lambda\phi = 0 & 1 < x < 2 \\ \phi(1) = 0,\ \phi(2) = 0 & \\ x^2\phi'' + 2x\phi' + \lambda\phi = 0 & \textit{A Cauchy-Euler Eq.} \end{array} \tag{30.1}$$

Let

$$\phi(x) = x^r \quad r(r-1) + 2r + \lambda = r^2 + r + \lambda = 0. \tag{30.2}$$

Therefore

$$r = \frac{-1 \pm \sqrt{1-4\lambda}}{2} = r_1, r_2. \tag{30.3}$$

Case I: $\boxed{\lambda = \mathbf{\tfrac{1}{4}}:}$

$$\phi(x) = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{1}{2}} \log x \tag{30.4}$$

$$\phi(1) = c_1 = 0 \ \phi(2) = c_2 2^{-\frac{1}{2}} \log 2 = 0 \Rightarrow c_2 = 0 \tag{30.5}$$

so there is no nontrivial eigenfunction for $\lambda = 1/4$.
Case II: $\boxed{\lambda \neq \mathbf{\tfrac{1}{4}}:}$

$$\phi(x) = c_1 x^{r_1} + c_2 x^{r_2} \tag{30.6}$$

$$\phi(1) = c_1 + c_2 = 0 \ c_2 = -c_1 \tag{30.7}$$

$$\phi(2) = c_1\big(2^{r_1} - 2^{r_2}\big) = 0 \tag{30.8}$$

$$\begin{array}{l} 2^{r_1 - r_2} = 1 \\ e^{(r_1 - r_2)\ln 2} = 1 = e^{2\pi i n} \\ (r_1 - r_2)\ln 2 = 2\pi i n \\ r_1 - r_2 = \sqrt{1-4\lambda} = 2\pi n i/\ln(2) \end{array} \tag{30.9}$$

Thus to obtain nontrivial solutions we require $1-4\lambda<0$ which implies $\boxed{\lambda>\frac{1}{4}}$. Thus for $\lambda>\frac{1}{4}$

$$\sqrt{1-4\lambda}=i\sqrt{4\lambda-1}=2\pi ni/\ln(2). \tag{30.10}$$

The Eigenvalues are:

$$\lambda_n=\frac{1}{4}+\frac{\pi^2n^2}{[\ln(2)]^2},\ 4\lambda_n-1=\frac{4\pi^2n^2}{[\ln(2)]^2}=(2\beta_n)^2\quad \beta_n=(n\pi/\ln 2). \tag{30.11}$$

The corresponding roots $r_1$ and $r_2$ are as follows

$$(r_1)_n=-\frac{1}{2}+i\beta_n \text{ and } (r_2)_n=-\frac{1}{2}-i\beta_n \tag{30.12}$$

$$\phi_n(x)=c_nx^{-\frac{1}{2}}\left(x^{i\beta_n}-x^{-i\beta_n}\right) \tag{30.13}$$

$$=c_nx^{-\frac{1}{2}}\left[e^{i\beta_n\ln x}-e^{-i\beta_n\ln x}\right] \tag{30.14}$$

$$=d_nx^{-\frac{1}{2}}\sin\left(\beta_n\ln x\right) \tag{30.15}$$

$$=d_nx^{-\frac{1}{2}}\sin\left[n\pi\frac{\ln x}{\ln(2)}\right] \tag{30.16}$$

Now let us consider expanding a function $f(x)$ in terms of a 'Fourier Series' of these new eigenfunctions in the following form

$$f(x)=\sum_{n=1}^{\infty}c_n\phi_n(x)=\sum_{n=1}^{\infty}c_nx^{-1/2}\sin\left(\frac{n\pi\ln x}{\ln 2}\right) \tag{30.17}$$

In order to determine the coefficients $c_n$ we project the function $f(x)$ onto the basis functions $\phi_n(x)$ as follows:

$$\int_1^2 f(x)x^{-1/2}\sin\left(\frac{m\pi\ln x}{\ln 2}\right)dx=\sum_{n=1}^{\infty}c_n\int_1^2 x^{-1/2}\sin\left(\frac{m\pi\ln x}{\ln 2}\right)x^{-1/2}\sin\left(\frac{n\pi\ln x}{\ln 2}\right)dx$$

The integrals under the summation can be evaluated by making the simple substitution $z=\ln x$ so that $dz=\frac{dx}{x}$, in which case:

$$\int_1^2\sin\left(\frac{m\pi\ln x}{\ln 2}\right)\sin\left(\frac{n\pi\ln x}{\ln 2}\right)\frac{dx}{x}=\int_0^{\ln 2}\sin\left(\frac{m\pi z}{\ln 2}\right)\sin\left(\frac{n\pi z}{\ln 2}\right)dz=\delta_{mn}\frac{\ln 2}{2}$$

Substituting this result into (30.17) we obtain the following expression for the Fourier coefficients $c_n$:

$$c_n=\frac{2}{\ln 2}\int_1^2 f(x)x^{-1/2}\sin\left(\frac{n\pi\ln x}{\ln 2}\right)dx$$

**Example 30.2** *A variable coefficient Heat Conduction Problem with Cauchy-Euler Eigenfunctions:*

$$\begin{array}{ll} u_t = D(x^2 u_x)_x - u & 1 < x < 2 \quad t > 0 \\ u(1,t) = 0 = u(2;t) & u(x,0) = f(x). \end{array} \tag{30.18}$$

Let

$$u(x,t) = X(x)T(t) \tag{30.19}$$

$$\frac{\dot{T}(t)}{DT(t)} = \frac{(x^2X')'}{X} - \frac{1}{D} \tag{30.20}$$

$$\frac{\dot{T}(t)}{DT(t)} + \frac{1}{D} = \frac{(x^2X')'}{X} = -\lambda \tag{30.21}$$

$$\begin{array}{ll} \dot{T} + (1 + D\lambda)T = 0 & T(t) = ce^{-(1+D\lambda)t} \\ \left.\begin{array}{r} (x^2X')' + \lambda X = 0 \\ X(1) = 0 = X(2) \end{array}\right\} & \begin{array}{l} \lambda_n = \frac{1}{4} + \frac{(\pi n)^2}{[\ln(2)]^2}; X_n(x) = x^{-\frac{1}{2}} \sin\left(n\pi \frac{\ln x}{\ln 2}\right) \\ n = 1, 2, \ldots \end{array} \end{array} \tag{30.22}$$

$$u(x,t) = x^{-\frac{1}{2}} \sum_{n=1}^{\infty} c_n e^{-(1+D\lambda_n)t} \sin\left(n\pi \frac{\ln x}{\ln 2}\right) \tag{30.23}$$

$$f(x) = u(x,0) = x^{-\frac{1}{2}} \sum_{n=1}^{\infty} c_n \sin\left(n\pi \frac{\ln x}{\ln 2}\right) \tag{30.24}$$

where

$$c_n = \frac{2}{\ln 2} \int_1^2 f(x) x^{-1/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) dx$$

### 30.1 Solving Laplace's equation using Cauchy-Euler eigenfunctions

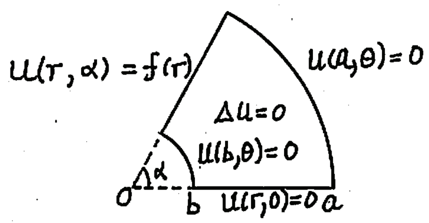


FIGURE 1. Annular sector subtending an arc of $\alpha$ radians between the radii $a$ and $b$: $1 = b < a = 2$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad 1 < r < 2,\ 0 < \theta < \alpha \tag{30.25}$$

$$u(r,0) = 0,\ u(r,\alpha) = f(r) \tag{30.26}$$

$$u(1,\theta) = 0, u(2,\theta) = 0 \tag{30.27}$$

$$u(r,\theta) = R(r)\Theta(\theta) \tag{30.28}$$

$$\frac{r^2R'' + rR'}{R(r)} = -\frac{\Theta''}{\Theta} = -\lambda^2 \quad \text{(because of Homog. BC)} \tag{30.29}$$

**Θ**: $\Theta'' - \lambda^2\Theta = 0 \qquad \Theta = C\cosh\lambda\theta + D\sinh\lambda\theta$
$\Theta(0) = 0 \qquad \Theta(0) = C = 0 \Rightarrow \Theta(\theta) = D\sinh\lambda\theta$

**R**: $r^2R'' + rR' + \lambda^2 R = 0(\star)$, $\quad R(1) = 0 = R(2)$ Although we can easily see that dividing through by $r$ we can reduce $(\star)$ to S-L form, let us use the integrating factor

$$\mu(r) = \frac{1}{P}e^{\int \frac{Q}{P}dr} = \frac{1}{r^2}e^{\int \frac{r}{r^2}dr} = \frac{1}{r^2}e^{\ln r} = \frac{1}{r}. \tag{30.30}$$

Therefore $-\frac{1}{r}\cdot(\star)$ and some rearrangement implies:

$$-rR'' - R' = -\left(rR'\right)' = \frac{\lambda^2}{r}R$$

Now let us look for Eigenvalues and Eigenfunctions to $(\star)$. Let

$$R(r) = r^\gamma \Rightarrow \gamma(\gamma-1) + \gamma + \lambda^2 = \gamma^2 + \lambda^2 = 0 \quad \gamma = \pm i\lambda. \tag{30.31}$$

$$R(r) = c_1 r^{i\lambda} + c_2 r^{-i\lambda} \quad r^{i\lambda} = e^{i\lambda\ln r} \tag{30.32}$$

$$= A\cos(\lambda\ln r) + B\sin(\lambda\ln r) \tag{30.33}$$

$$R(1) = A\cos\left[\lambda(\ln 1)\right] + B\sin(\lambda\ln 1) = A = 0 \tag{30.34}$$

$$R(2) = B\sin\left[\lambda\ln 2\right] = 0 \Rightarrow \lambda_n\ln 2 = n\pi \quad n = 1, 2, \ldots \tag{30.35}$$

and the corresponding Eigenfunctions are $R_n = \sin\left(n\pi\frac{\ln r}{\ln 2}\right)$. Therefore

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n\sinh(\lambda_n\theta)\sin(\lambda_n\ln r). \tag{30.36}$$

Now match BC:

$$f(r) = u(r,\alpha) = \sum_{n=1}^{\infty} B_n\sinh(\lambda_n\alpha)\sin\left(n\pi\frac{\ln r}{\ln 2}\right) \tag{30.37}$$

Now making the substitution $x = \ln r$ so that $dx = \frac{dr}{r}$ we can reduce the orthogonality integrals for the Fourier coefficients to the form:

$$\int_1^2 \frac{1}{r}\sin\left(\frac{m\pi\ln r}{\ln 2}\right)\sin\left(\frac{n\pi\ln r}{\ln 2}\right)\,dr = \int_0^{\ln 2}\sin\left(\frac{m\pi x}{\ln 2}\right)\sin\left(\frac{n\pi x}{\ln 2}\right)\,dx = \begin{cases} 0 & m \neq n \\ \frac{\ln 2}{2} & m = n \end{cases}. \tag{30.38}$$

Therefore

$$B_n = \frac{2}{\ln 2\sinh\left(\frac{n\pi\alpha}{\ln 2}\right)}\int_1^2 \frac{f(r)}{r}\sin\left(\frac{n\pi\ln r}{r}\right)\,dr. \tag{30.39}$$