Complex form of Fourier Series:

Recall that the Fourier ~~transform~~ series of a function $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{--- } (*)$$

Recall that we can write

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{--- } (1)$$

$$e^{-i\theta} = \cos\theta + -i\sin\theta \quad \text{--- } (2)$$

Add (1) and (2) to get

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

subtracting (1) and (2) leads to

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Let $\theta = \frac{n\pi x}{L}$, we get

$$\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\left(\frac{n\pi x}{L}\right)} + e^{-i\left(\frac{n\pi x}{L}\right)}}{2}$$

and

$$\sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\left(\frac{n\pi x}{L}\right)} - e^{-i\left(\frac{n\pi x}{L}\right)}}{2i}$$

Substituting this into * we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{i\left(\frac{n\pi x}{L}\right)} + e^{-i\left(\frac{n\pi x}{L}\right)}\right)$$

$$+ \frac{b_n}{2i}\left(e^{i\left(\frac{n\pi x}{L}\right)} - e^{-i\left(\frac{n\pi x}{L}\right)}\right)$$

$\boxed{\text{Note } \frac{b_n}{i}\cdot\frac{i}{i} = -b_n i}$

Combining terms

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{i\left(\frac{n\pi x}{L}\right)} + \frac{a_n + ib_n}{2} e^{-i\left(\frac{n\pi x}{L}\right)}$$

$*\alpha$

Let $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n - ib_n}{2}$

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{a_{-n} - ib_{-n}}{2}$$

Note 2

$$a_{-n} = \int_{-L}^{L} f(x)\cos\left(\frac{-n\pi x}{L}\right)dx = \int_{L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = a_n$$

$$b_{-n} = \int_{-L}^{L} f(x)\sin\left(\frac{-n\pi x}{L}\right)dx = -\int_{-L}^{L} f(x)\sin\left(\frac{-n\pi x}{L}\right)dx = -b_n$$

Rewrite the complex form (**) by expanding n from $-\infty$ to ∞ to have

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n\pi x}{L}\right)}, \quad C_n = \frac{a_n - ib_n}{2}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\cdot e^{-i\left(\frac{n\pi x}{L}\right)}dx$$

Therefore we can write the complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n\pi x}{L}\right)}$$ where

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi x}{L}\right)} dx$$

Example

$$f(x) = \begin{cases} \pi & -\pi \le x < 0 \\ 0 & 0 \le x < \pi \end{cases}$$

Find the Complex form of Fourier Serie of $f(x)$ with period 2

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{2}\right)}$$

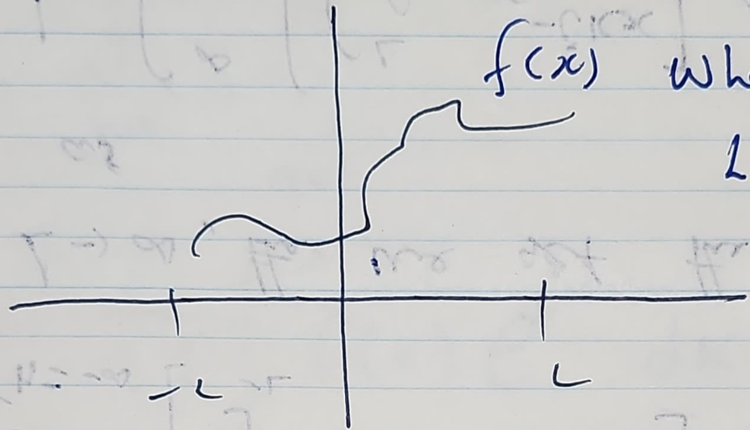
$$c_n = \frac{1}{2\cdot L} \int_{-L}^{L} f(x)\, e^{-i\left(\frac{n\pi x}{2}\right)} dx \quad , L = 1$$

$$= \frac{1}{2} \int_{-1}^{0} \pi e^{\frac{-in\pi x}{2}} dx = \frac{1}{\cancel{2}} \cdot \frac{\cancel{\pi}\cdot \cancel{2}}{-n\cancel{\pi} i} e^{\frac{-in\pi x}{2}} \Big|_{-1}^{0}$$

$$\frac{i}{n}\left\{ 1 - e^{\frac{in\pi}{2}} \right\}$$

Fourier transform

Fourier transform can be obtained from the complex Fourier series by taking the limit as $L \to \infty$ - That is



What happens as $L \to \infty$?

Consider the Fourier (complex) expansion of $f(x)$ periodic on the interval $[-L, L]$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)}$$

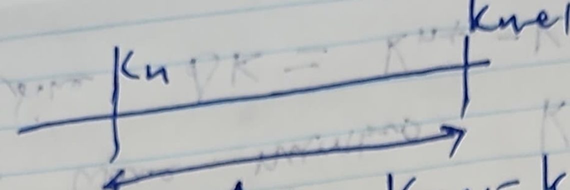
with $c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi x}{L}\right)} dx$

Therefore we can write -

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\left(\frac{n\pi x}{L}\right)} dx \right] e^{i\left(\frac{n\pi x}{L}\right)}$$

Define the wave number $k_n = \frac{n\pi}{L}$

and define $\Delta k = k_{n+1} - k_n$



$$\Delta k = k_{n+1} - k_n$$

$$= \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

as $L \to \infty$, $\Delta k \to dk = \frac{\pi}{L}$

We can now rewrite the Complex Fourier series as

$$f(x) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\underbrace{L} \cdot \frac{\pi}{\pi}}_{= dk} \int_{-L}^{L} f(x) e^{-ik_n x} dx \right] e^{ik_n x}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-L}^{L} f(x) e^{-ik_n x} dx \right] e^{ik_n x} \Delta k$$

as $L \to \infty$, ~~the~~ we get the Riemann sum written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left[\int_{-L}^{L} f(x) e^{-ikx} dx \right]}_{\hat{f}(k)} e^{ikx} dk$$

We obtain the Fourier transform pair

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) \cdot e^{-ikx} \, dx$$ Fourier transform

and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) \, e^{ikx} \, dk$$

$\rightarrow$ Inverse Fourier transform

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2} \end{cases}$$