Chapter 1

Review of techniques to solve Ordinary Differential Equations

The aim of this lecture is to provide you with a warm-up on some techniques for solving Ordinary Differential Equations (ODEs). You are likely already familiar with most of these techniques, as they were introduced in Math 215, 255, and 256. We will briefly review some of them which we will require throughout this course.

1.1 Separable equations

Definition 1.1.1 (Separable ODE)**.** *We say that a first-order ODE is separable if it can be written on the form:*

$$\frac{dy}{dx} = P(x)Q(y) \tag{1.1}$$

where P is a function of x only and Q is a function of y only.

Methodology 1.1.1. *To solve a separable ODE of the form (1.1), we proceed as follows:*

i. Bring all terms involving x to one side, and all terms involving y and dy to the other side.

ii. Integrate both sides with respect to their respective variables (don't forget an arbitrary constant).

iii. Solve the resulting equation for $y(x)$.

NOTE: When it is clear from the context, we may sometimes write y' to denote $\frac{dy}{dx}$.

Example 1.1.1. *Solve the following ODE, where $(x \neq 0)$:*

$$x^3 y' = e^{3y}$$

Solution 1.1.1 (Solution to Example 1.1.1). *We can separate the variables because $y' = \frac{dy}{dx}$, which gives*

$$e^{-3y}dy = \frac{1}{x^3}dx$$

By integrating, we obtain

$$\frac{e^{-3y}}{3} = \frac{1}{2x^2} + C$$

Thus,

$$y = -\frac{1}{3}\ln\left|\frac{3}{2x^2} + B\right|,$$

where $B \in \mathbb{R}$.

Example 1.1.2. *Solve the following ODE:*

$$y' = y\sin x$$

Solution 1.1.2 (Solution to Example 1.1.2). *We first separate variables:*

$$\frac{dy}{y} = \sin x\,dx$$

Integrating both sides gives:

$$\int\frac{dy}{y} = \int\sin x\,dx + C \quad \Rightarrow \quad \ln|y| = -\cos x + C$$

Thus, the solution is:

$$y(x) = Ae^{-\cos x}, \quad \text{where } A = e^C \in \mathbb{R}\setminus\{0\}.$$

Example 1.1.3. *Solve the following ODE:*

$$y' = y\cos x$$

Solution 1.1.3 (Solution to Example 1.1.3). *Separating the variables, we have:*

$$\frac{dy}{y} = \cos x\,dx$$

Integrating both sides:

$$\int\frac{dy}{y} = \int\cos x\,dx + C \quad \Rightarrow \quad \ln|y| = \sin x + C$$

Therefore, the solution is:

$$y(x) = Be^{\sin x}, \quad \text{where } B \in \mathbb{R}\setminus\{0\}.$$

Example 1.1.4. *Solve the following differential equation:*

$$\frac{dy}{dx} = 2x(y^2+1)$$

Solution 1.1.4 (Solution to Example 1.1.4). *We rewrite the equation to separate the variables x and y:*

$$\frac{1}{y^2+1}dy = 2x\,dx$$

Integrating both sides yields:

$$\int \frac{1}{y^2+1}dy = \int 2x\,dx + C \quad \Rightarrow \quad \arctan(y) = x^2 + C$$

To solve explicitly for y, we take the tangent of both sides:

$$y = \tan(x^2+C), \quad \textit{where } C \in \mathbb{R} \textit{ is an arbitrary constant.}$$

1.2 Linear first-order equations

First-order linear differential equations are the most interesting because we encounter them in different real-world applications, for example in biological applications, inphysics (electricity, mechanics, etc.). These are equations where y and y' are of the first degree.

Definition 1.2.1 (First-order linear differential equations)**.** *First-order linear differential equations take the general form*

$$a(x)y' + b(x)y = c(x) \tag{1.2}$$

where a, b, and c are functions of x.
If $a(x) \neq 0$, then Equation (1.2) can be written in standard form as :

$$y' + P(x)y = Q(x) \tag{1.3}$$

where $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$

NOTE: If $c(x) = 0$ (equivalently $Q(x) = 0$), then the equation obtained will be homogeneous, called a **homogeneous linear ODE**.

Methodology 1.2.1. *To solve the first-order linear ODE (1.3), we use the integrating factor method. Conisder*

$$Ly = y' + P(x)y = Q(x)$$

Multiply by an integrating factor $\mu(x)$:

$$\mu(x)Ly = \mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

Compare to:

$$(\mu(x)y(x))' = \mu' y + \mu y'$$

Choose:

$$\begin{aligned}\mu' y &= \mu p(x) y \\ \frac{d\mu}{\mu} &= p(x)dx \\ \int \frac{d\mu}{\mu} &= \int p(x)dx + C \\ \ln|\mu| &= \int p(x)dx + C \\ \mu =& e^{\int p(x)dx} \cdot e^c = c_0 e^{\int p(x)dx}\end{aligned}$$

The integrating factor is therefore,

$$\boxed{\mu = e^{\int p(x)\,dx}}$$

$$\begin{aligned}c_0 e^{\int p(x)\cdot dx} \cdot Ly &= c_0 e^{\int p(x)dx} y' + p(x) c_0 e^{\int p(x)dx} y \\ &= c_0 e^{\int p(x)dx} Q(x) \\ \to \left(e^{\int p(x)dx} y\right)' &= e^{\int p(x)dx} Q(x) \\ e^{\int p(x)dx} y &= \int^x e^{\int p(t)dt} Q(s)ds + C \\ y(x) =& e^{-\int p(x)dx} \cdot \int^x e^{\int p(t)dt} Q(s)ds + C\, e^{-\int p(x)dx}\end{aligned}$$

Example 1.2.1. *Solve the following differential equation:*

$$x\frac{dy}{dx} + 2y = 10x^2$$

Solution 1.2.1 (Solution to Example 1.2.1). *For $x \neq 0$, we rewrite the equation in standard linear form:*

$$\frac{dy}{dx} + \frac{2}{x}y = 10x$$

This is now in the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x) = \frac{2}{x}$ and $q(x) = 10x$. The integrating factor $\mu(x)$ is given by:

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{2}{x}\,dx} = x^2$$

Now, we multiply the differential equation by $\mu(x) = x^2$:

$$x^2\frac{dy}{dx} + 2xy = 10x^3$$

The left-hand side is the derivative of a product:

$$\frac{d}{dx}(x^2y) = 10x^3$$

Integrating both sides with respect to x:

$$\int \frac{d}{dx}(x^2y)\,dx = \int 10x^3\,dx + C$$

we obtain:

$$x^2y = \frac{5x^4}{2} + C$$

Solving for y:

$$y = \frac{5x^2}{2} + \frac{C}{x^2}, \quad x \neq 0$$

where $C \in \mathbb{R}$ is an arbitrary constant.

Example 1.2.2. *Solve the following differential equation:*

$$\frac{dy}{dx} + \tan(x)y = \sin(x)$$

Solution 1.2.2 (Solution to Example 1.2.2). *This is a linear ODE with $p(x) = \tan(x)$ and $q(x) = \sin(x)$. The integrating factor is:*

$$\mu(x) = e^{\int \tan(x)\,dx} = e^{-\ln|\cos(x)|} = \frac{1}{\cos(x)}$$

Multiplying both sides of the differential equation by $\mu(x)$:

$$\frac{1}{\cos(x)}\frac{dy}{dx} + \frac{\tan(x)}{\cos(x)}y = \frac{\sin(x)}{\cos(x)}$$

This simplifies to:

$$\frac{d}{dx}\left(\frac{y}{\cos(x)}\right) = \tan(x)$$

Integrating both sides:

$$\int \frac{d}{dx}\left(\frac{y}{\cos(x)}\right)dx = \int \tan(x)dx + C$$

$$\frac{y}{\cos(x)} = -\ln|\cos(x)| + C \quad \Rightarrow \quad y = -\cos(x)\ln|\cos(x)| + C\cos(x)$$

Example 1.2.3. *Solve the following differential equation:*

$$\frac{dy}{dx} + \cot(x)y = \cos(x)$$

Solution 1.2.3 (Solution to Example 1.2.3). *This is a linear ODE with* $p(x) = \cot(x)$ *and* $q(x) = \cos(x)$*. The integrating factor is:*

$$\mu(x) = e^{\int \cot(x)\,dx} = e^{\ln|\sin(x)|} = \sin(x)$$

Multiplying both sides of the differential equation by $\mu(x) = \sin(x)$*:*

$$\sin(x)\frac{dy}{dx} + \cos(x)y = \cos(x)\sin(x)$$

The left-hand side is the derivative of:

$$\frac{d}{dx}(y\sin(x)) = \cos(x)\sin(x)$$

Integrating both sides:

$$\int \frac{d}{dx}(y\sin(x))dx = \int \cos(x)\sin(x)dx + C$$

We use the identity $\cos(x)\sin(x) = \frac{1}{2}\sin(2x)$*, so:*

$$y\sin(x) = \frac{-1}{4}\cos(2x) + C \quad \Rightarrow \quad y = \frac{-\cos(2x)}{4\sin(x)} + \frac{C}{\sin(x)}$$

1.3 Second-order linear homogeneous equations with constant coefficients

Definition 1.3.1 (Homogeneous second-order ODE)**.** *We call second-order linear homogeneous differential equation an ODE of the form:*

$$Ly := ay'' + by' + cy = 0 \tag{1.4}$$

where $a, b,$ *and* c *are real constants.*

To solve this type of equation, we assume a solution of the form:

$$y(x, r) = e^{rx}$$

where r is a constant to be determined. Substituting this expression for $y(x)$ into the differential equation gives the corresponding characteristic equation:

$$\boxed{\underbrace{ar^2 + br + c = 0}_{\textbf{characteristic equation}}}$$

Solving this quadratic equation yields two possible values for r :

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Depending on the discriminant $\Delta = b^2 - 4ac$, we consider different possibilities as shown below;

1. **Distinct real roots:** If discrimant $\Delta > 0$, the characteristic polynomial gives two distinct roots r_1 and r_2. The general solution to the differential equation is:

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x},}$$

   where c_1 and c_2 are arbitrary constants.

2. **Complex roots:** If $\Delta < 0$, r_1 and r_2 are complex conjugates, which can be written in the form;

$$r_1 = \alpha + i\beta \quad \text{and} \quad r_2 = \alpha - i\beta,$$

   where α and β are real constants, the general solution to the differential equation can be written as:

$$y(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$

   Using Euler's formula $e^{(\alpha \pm i\beta)x} = e^{\alpha x}[\cos(\beta x) \pm i \sin(\beta x)]$, the solution can also be expressed as:

$$\boxed{y(x) = e^{\alpha x}[C_1 \cos(\beta x) + C_2 \sin(\beta x)],}$$

   where C_1 and C_2 are arbitrary constants. This form is typically preferred in practice since it involves real-valued functions.

3. **Repeated real roots** If $\Delta = 0$, we obtain two equal real roots such that $r_1 = r_2 = -\frac{b}{2a}$. We obtain

$$y_1(x) = e^{-\frac{b}{2a}x}$$

   is one solution. We need a second solution, $y_2(x)$ to build the general solution.

   How do we find $y_2(x)$?

   Remember from ODE class, we guessed:

$$y_2(x) = x e^{r_1 x}.$$

   A better way to find $y_2(x)$ that leads to this guess is:

$$\begin{aligned}
y(r, x) &= e^{rx} \\
Ly(x, r) &= \left(ar^2 + br + c\right) e^{rx} \\
&= a\left[r^2 + \frac{b}{a}r + \frac{c}{a}\right] e^{rx} \\
&= a\left[\left(r + \frac{b}{2a}\right)^2 - \underbrace{\left(\frac{b^2 - 4ac}{4a^2}\right)}_{(\Delta=0)}\right] e^{rx} \\
&= a\left[\left(r + \frac{b}{2a}\right)^2\right] e^{rx} = a\left(r - r_1\right)^2 e^{rx}
\end{aligned}$$

Now, take partial derivative with respect to r :

$$\begin{aligned}\frac{\partial}{\partial r} L y(x, r) &= 2a\left(r-r_1\right) e^{rx}+x\left(r-r_1\right)^2 e^{rx} \\ &= 0 \quad \text{if} \quad r=r_1\end{aligned}$$

We also know that

$$\frac{\partial}{\partial r} L y(x, r)=L\left[\frac{\partial}{\partial r} y(x, r)\right]_{r=r_1}=0 \quad \text{if } r=r_1$$

So, if $\left.y(x, r)\right|_{r=r,}$ is a solution, $\left.\frac{\partial}{\partial r} y(x, r)\right|_{r=r,}$ is also a solution and we have found the second solution:

$$y_2(x)=\left.\frac{\partial}{\partial r} y(x, r)\right|_{r=r_1}=\left.\frac{\partial}{\partial r} e^{rx}\right|_{r=r_1}=x e^{r_1 x}$$

$y_1(x)=e^{r_1 x}$ and $y_2(x)=x e^{r_1 x}$ are linearly independent and the general solution is:

$$\boxed{y(x)=c_1 e^{r_1 x}+c_2 x e^{r_1 x},}$$

Example 1.3.1. *Solve the initial value problem:*

$$y''+2y'+5y=0, \quad y(0)=0, \quad y'(0)=1$$

Solution 1.3.1 (Solution to Example 1.3.1). *Assume $y=e^{rx}$, then the characteristic equation is:*

$$r^2+2r+5=0$$

Solving using the quadratic formula:

$$r=\frac{-2 \pm \sqrt{(-2)^2-4(1)(5)}}{2(1)}=-1 \pm 2i$$

The general solution is:

$$y(x)=e^{-x}\left(c_1 \cos (2x)+c_2 \sin (2x)\right)$$

Apply initial conditions:

$$\begin{aligned} & y(0)=e^0\left(c_1 \cdot 1+c_2 \cdot 0\right)=c_1=0 \\ & y'(x)=\frac{d}{dx}\left[e^{-x}\left(c_1 \cos (2x)+c_2 \sin (2x)\right)\right] \\ & \quad=e^{-x}\left[-\left(c_1 \cos (2x)+c_2 \sin (2x)\right)+c_2 \cdot 2 \cos (2x)-c_1 \cdot 2 \sin (2x)\right] \\ & y'(0)=1 \Rightarrow e^0\left[-\left(0 \cdot 1+c_2 \cdot 0\right)+c_2 \cdot 2 \cdot 1-0 \cdot 0\right]=2c_2=1 \Rightarrow c_2=\frac{1}{2}\end{aligned}$$

Final solution:

$$y(x)=\frac{1}{2} e^{-x} \sin (2x)$$

Example 1.3.2. *Solve the following differential equation:*

$$y'' - 9y = 0$$

Solution 1.3.2 (Solution to Example 1.3.2). *Assume a trial solution of the form $y = e^{rx}$, then the characteristic equation becomes:*

$$r^2 - 9 = 0$$

Solving for r:

$$r^2 = 9 \quad \Rightarrow \quad r = \pm 3$$

Therefore, the general solution is:

$$y(x) = c_1 e^{3x} + c_2 e^{-3x},$$

where c_1 and c_2 are arbitrary constants.

Example 1.3.3. *Solve the following differential equation:*

$$y'' + 9y = 0$$

Solution 1.3.3 (Solution to Example 1.3.3). *Assume $y = e^{rx}$, then the characteristic equation is:*

$$r^2 + 9 = 0$$

Solving for r:

$$r^2 = -9 \quad \Rightarrow \quad r = \pm 3i$$

The general solution is:

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x),$$

where c_1 and c_2 are arbitrary constants.

Example 1.3.4. *Solve the following differential equation:*

$$y'' + 6y' + 9y = 0$$

Solution 1.3.4 (Solution to Example 1.3.4). *Assume $y = e^{rx}$, then the characteristic equation is:*

$$r^2 + 6r + 9 = 0$$

Factorizing:

$$(r + 3)^2 = 0 \quad \Rightarrow \quad r = -3$$

The general solution is:

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x},$$

where c_1 and c_2 are arbitrary constants.

Example 1.3.5. *Solve the following differential equation:*

$$y'' + 6y' - 9y = 0$$

Solution 1.3.5 (Solution to Example 1.3.5). *Assume $y = e^{rx}$, then the characteristic equation is:*

$$r^2 + 6r - 9 = 0$$

Solving using the quadratic formula:

$$r = \frac{-6 \pm \sqrt{6^2 - 4(1)(-9)}}{2(1)} = -3 \pm 3\sqrt{2}$$

The general solution is:

$$y(x) = c_1 e^{(-3+3\sqrt{2})x} + c_2 e^{(-3-3\sqrt{2})x},$$

where c_1 and c_2 are arbitrary constants.

1.4 Second-order Euler equations/Equidimensional equations

Definition 1.4.1. *A second-order Euler equation is a type of differential equation written as:*

$$x^2 y'' + \alpha x y' + \beta y = 0$$

where α, and β are constants.

To solve this equation, we guess a solution of the form:

$$y(x, r) = x^r$$

where r is a number we need to find. Substitute to find

$$\begin{aligned} Ly(x,r) &= r(r-1)x^r + \alpha r x^r + \beta x^r \\ &= \left[r^2 + (\alpha - 1)r + \beta\right] x^r = 0 \end{aligned}$$

$$x^r \neq 0 \to \underbrace{r^2 + (\alpha - 1)r + \beta = 0}_{\text{Characteristic equation}}$$

This gives the roots

$$r_{1,2} = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2},$$

where

$$\Delta = (\alpha - 1)^2 - 4\beta \quad \text{is the discriminant}$$

Three different cases can happen:

Case 1: Distinct real roots $(\Delta > 0)$ If

$$\Delta = (\alpha - 1)^2 - 4\beta > 0,$$

the indicial equation has two distinct real roots r_1 and r_2. The general solution is

$$\boxed{y(x) = c_1 x^{r_1} + c_2 x^{r_2},}$$

where c_1, c_2 are arbitrary constants.

Note: If either root is negative then $|y| \to \infty$ as $x \to 0$.

Case 2: Repeated real roots $(\Delta = 0)$ If

$$\Delta = (\alpha - 1)^2 - 4\beta = 0,$$

there is a double root r_1. One finds

$$y_1(x) = x^{r_1}, \qquad y_2(x) = x^{r_1} \ln x,$$

so the general solution is

$$\boxed{y(x) = \big(c_1 + c_2 \ln x\big)\, x^{r_1}.}$$

Note here that $y_2(x)$ has been found in the same way as for the second order ODE with constant coefficients.

Case 3: Complex conjugate roots $(\Delta < 0)$ If

$$\Delta = (\alpha - 1)^2 - 4\beta < 0,$$

then

$$r_{\pm} = \frac{1-\alpha}{2} \pm i\,\frac{\sqrt{4\beta - (\alpha-1)^2}}{2} = \lambda \pm i\mu.$$

Writing $\lambda = \frac{1-\alpha}{2}$, $\mu = \frac{\sqrt{4\beta-(\alpha-1)^2}}{2}$, the general real solution is

$$\boxed{y(x) = x^{\lambda}\Big(A_1 \cos(\mu \ln x) + A_2 \sin(\mu \ln x)\Big).}$$

For $x < 0$, replace x by $|x|$ in the ln.

Example 1.4.1. *Solve the initial value problem:*

$$x^2 y'' - x y' + 3y = 0, \quad y(1) = 0, \quad y'(1) = 1$$

Solution 1.4.1 (Solution to Example 1.4.1). *Assume a solution of the form $y = x^r$, substitute into the differential equation:*

$$x^2(r(r-1)x^{r-2}) - x(rx^{r-1}) + 3x^r = 0$$

Simplify:

$$r(r-1)x^r - rx^r + 3x^r = (r^2 - 2r + 3)x^r = 0$$

Characteristic equation:

$$r^2 - 2r + 3 = 0 \Rightarrow r = 1 \pm i$$

General solution:

$$y(x) = x\left[c_1 \cos(\ln x) + c_2 \sin(\ln x)\right]$$

Apply initial conditions:

$$\begin{aligned}
y(1) &= 1 \cdot (c_1 \cdot 1 + c_2 \cdot 0) = c_1 = 0 \\
y'(x) &= \frac{d}{dx}\left[x(c_1 \cos(\ln x) + c_2 \sin(\ln x))\right] \\
&= c_1 \cos(\ln x) + c_2 \sin(\ln x) + x \cdot \left[-c_1 \sin(\ln x) + c_2 \cos(\ln x)\right] \cdot \frac{1}{x} \\
&= c_1 \cos(\ln x) + c_2 \sin(\ln x) - c_1 \sin(\ln x) + c_2 \cos(\ln x) \\
y'(1) &= 0 + c_2 \cdot 1 = 1 \Rightarrow c_2 = 1
\end{aligned}$$

Final solution:

$$y(x) = x \sin(\ln x)$$

Example 1.4.2. *Solve the initial value problem:*

$$x^2 y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 1$$

Solution 1.4.2 (Solution to Example 1.4.2). *Assume $y = x^r$, substitute into the equation:*

$$r(r-1)x^{r-2} - 3rx^{r-1} + 4x^r = 0$$

Multiply through by x^2:

$$r(r-1) - 3r + 4 = 0 \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$$

Double root: $r = 2$

$$y(x) = c_1 x^2 + c_2 x^2 \ln x$$

Apply initial conditions:

$$\begin{aligned}
y(1) &= c_1 + c_2 \cdot 0 = 1 \Rightarrow c_1 = 1 \\
y'(x) &= 2c_1 x + c_2(2x \ln x + x) \\
y'(1) &= 2 + c_2 = 1 \Rightarrow c_2 = -1
\end{aligned}$$

Final solution:

$$y(x) = x^2 - x^2 \ln x$$

Example 1.4.3. *Solve the following differential equation:*

$$x^2y'' + xy' - 9y = 0$$

Solution 1.4.3 (Solution to Example 1.4.3). *Assume a solution of the form $y = x^r$, then substitute into the equation:*

$$x^2(r(r-1)x^{r-2}) + x(rx^{r-1}) - 9x^r = 0$$

Simplify:

$$r(r-1)x^r + rx^r - 9x^r = 0 \Rightarrow r^2 - 9 = 0$$

Solving for r:

$$r = \pm 3$$

The general solution is:

$$y(x) = c_1x^3 + c_2x^{-3}$$

where c_1 and c_2 are arbitrary constants.

Example 1.4.4. *Solve the following initial value problem:*

$$x^2y'' - 3xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 0$$

Solution 1.4.4 (Solution to Example 1.4.4). *Assume $y = x^r$, substitute into the equation:*

$$r(r-1)x^{r-2} - 3rx^{r-1} + 4x^r = 0$$

Multiply through by x^2:

$$r(r-1) - 3r + 4 = 0 \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$$

Since $r = 2$ is a double root, the general solution is:

$$y(x) = c_1x^2 + c_2x^2 \ln x$$

Apply initial conditions:

$$y(1) = c_1 + c_2 \cdot 0 = 1 \Rightarrow c_1 = 1$$
$$y'(x) = \frac{d}{dx}[c_1x^2 + c_2x^2 \ln x] = 2c_1x + c_2(2x \ln x + x)$$
$$y'(1) = 2 \cdot 1 + c_2(0 + 1) = 2 + c_2 = 0 \Rightarrow c_2 = -2$$

Final solution:

$$y(x) = x^2 - 2x^2 \ln x$$

If $x < 0$, replace x with $|x|$ in the logarithm.