Chapter 2

Series solutions of variable coefficient ordinary differential equations

In this part, we introduce a method to solve linear differential equations with variable coefficients, which frequently arise in physical problems. They do not usually admit solutions expressible in terms of elementary functions. Such equations can often be solved using numerical methods, but in many cases, it is easier to find solutions in the form of an infinite series. We introduce the concepts of ordinary points about which Taylor series solutions are obtained and singular points about which more general solutions are required.

2.1 Power series method

The power series method is the standard method for solving linear ODEs with variable coefficients. It gives solutions in the form of power series. These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions, as we shall see. In this section we begin by explaining the idea of the power series method.

Definition 2.1.1: Power series

Definition 2.1.1. *A power series (in powers of $x-x_0$) is an infinite series of the form*

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \cdots + a_n (x-x_0)^n + \cdots \tag{2.1}$$

Here, x is a variable. $a_0, a_1, a_2, \cdots$ are constants, called the coefficients of the series. x_0 is a constant, called the center of the series.

In particular, if $x_0 = 0$, we obtain a power series in powers of x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{2.2}$$

We shall assume that all variables and constants are real.

Example 2.1.1. *Familiar examples of power series are the Taylor-Maclaurin series*

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad (|x| < 1, \text{ geometric series })$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

More generally, if we know all the derivatives of a function $f(x)$ at a single point x_0, then we have the Taylor approximation:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

for x near x_0. We note that the term "power series" usually refers to a series of the form (2.1) (or (2.2)) but does not include series of negative or fractional powers of x.

Before moving further, we review relevant properties of power series.

Definition 2.1.2. *The power series* (2.1) *is said to converge for a given x if the limit*

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n$$

exists. Otherwise, the series diverges for the given x.
For any power series (2.1), *exactly one of the following statements is true:*

(a) The power series converges only for $x = x_0$.

(b) The power series converges for all values of x.

(c) There is a positive number R such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.

In case (c), R is called the radius of convergence of the power series. For convenience, we include the other two cases in this definition by setting $R = 0$ in case (a) and $R = \infty$ in case (b). The open interval of convergence is defined as:

$$(x_0 - R, x_0 + R) \quad \text{if } 0 < R < \infty, \quad \text{or} \quad (-\infty, \infty) \quad \text{if } R = \infty$$

If R is finite, no general statement can be made about convergence at the endpoints $x = x_0 \pm R$. The series may converge at one or both endpoints, or diverge at both.

Theorem 2.1.1 (Ratio test)**.** *Consider the series* $\sum_{n=0}^{\infty} c_n$ *and suppose:*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

Then:

(a) If $L < 1$, *then* $\sum_{n=0}^{\infty} c_n$ *converges.*

(b) If $L > 1$ *or the limit approaches* ∞, *then* $\sum_{n=0}^{\infty} c_n$ *diverges.*

(c) If $L = 1$, *the ratio test is inconclusive, and another test must be used.*

The following examples help us understand Theorem 2.1.1.

Example 2.1.2. *Find the radius of convergence for the following series:*

$$\sum_{n=0}^{\infty} n!x^n$$

Solution 2.1.1 (Solution to Example 2.1.2). *Here,* $c_n = n!x^n$. *Using the ratio test:*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

Thus, the series converges only at $x = 0$, *so* $R = 0$.

Example 2.1.3. *Find the radius of convergence for the following series:*

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$$

Solution 2.1.2 (Solution to Example 2.1.3). *Here,* $c_n = \frac{(-1)^n x^n}{n!}$. *Using the ratio test:*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = |x| \cdot \lim_{n\to\infty}\frac{1}{n+1} = 0$$

Hence, the series converges for all x, *and* $R = \infty$.

Example 2.1.4. *Find the radius of convergence for the following series:*

$$\sum_{n=0}^{\infty} \frac{2^n}{n^2}(x-1)^n$$

Solution 2.1.3 (Solution to Example 2.1.4). *Here,* $c_n = \frac{2^n}{n^2}(x-1)^n$. *Using the ratio test:*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+1}/(n+1)^2}{2^n(x-1)^n/n^2}\right| = 2|x-1|\cdot\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = 2|x-1|$$

The series converges when $2|x-1| < 1$, *or* $|x-1| < \frac{1}{2}$. *Thus,* $R = \frac{1}{2}$.

Example 2.1.5. *Find the radius of convergence for the following series:*

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{8^m}x^{3m}$$

Solution 2.1.4 (Solution to Example 2.1.5). *Here,* $c_n = \frac{(-1)^m}{8^m}x^{3m}$. *Using the ratio test:*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \frac{|x|^3}{8}$$

The series converges when $\frac{|x|^3}{8} < 1$, *or* $|x| < 2$. *Thus,* $R = 2$.

2.1.1 Idea of the power series method

The idea of the power series method for solving ODEs is simple and natural. We describe the practical procedure and illustrate it for two ODEs whose solution we know, so that we can see what is going on.

Important note !

For a given ODE

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2.3}$$

we first represent $P(x), Q(x)$ and $R(x)$ by power series in powers of x (or of $x - x_0$ if solutions in powers of $x - x_0$ are wanted). Next we assume a solution in the form of a power series with unknown coefficients,

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{2.4}$$

Substitute back this series and the series obtained by termwise differentiation,

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \tag{2.5}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + \cdots \tag{2.6}$$

into the ODE (2.3) . Then we collect like powers of x and equate the sum of the coefficients of each occurring power of x to zero, starting with the constant terms, then taking the terms containing x, then the terms in x^2, and so on. This gives equations from which we can determine the unknown coefficients of (2.4) successively.

Remark 2.1.1. *Note that if the power series (2.4) has a positive radius of convergence R, then the radius of convergence of all its successive derivative series is also R.*

Let us show this for three simple ODEs that can also be solved by elementary methods, so that we would not need power series.

Example 2.1.6. *Solve the following ODE by power series:*

$$y' + 2y = 0$$

Solution 2.1.5 (Solution to Example 2.1.6). *We substitute the power series expansions:*

$$\sum_{m=1}^{\infty} m a_m x^{m-1} + 2\sum_{m=0}^{\infty} a_m x^m = 0$$

Change index to match powers of x:

$$\sum_{s=0}^{\infty}(s+1)a_{s+1}x^s + 2\sum_{s=0}^{\infty} a_s x^s = 0$$

Combine terms:

$$\sum_{s=0}^{\infty}\left[(s+1)a_{s+1} + 2a_s\right]x^s = 0$$

Set each coefficient to zero:

$$(s+1)a_{s+1} + 2a_s = 0 \quad \Rightarrow \quad a_{s+1} = -\frac{2a_s}{s+1}$$

Compute terms:

$$a_1 = -2a_0, \quad a_2 = \frac{2^2 a_0}{2!}, \quad a_3 = -\frac{2^3 a_0}{3!}, \ldots$$

General term:

$$a_s = (-1)^s \frac{2^s a_0}{s!}$$

Thus:

$$y(x) = \sum_{s=0}^{\infty}(-1)^s \frac{2^s a_0}{s!} x^s = a_0 e^{-2x}$$

Example 2.1.7. *Solve the following ODE by power series:*

$$y'' + y = 0$$

Solution 2.1.6 (Solution to Example 2.1.7). *Substitute series expansions:*

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

Change indices:

$$\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2}x^s + \sum_{s=0}^{\infty} a_s x^s = 0$$

Combine:

$$\sum_{s=0}^{\infty} \left[(s+2)(s+1)a_{s+2} + a_s\right] x^s = 0$$

Recurrence relation:

$$a_{s+2} = -\frac{a_s}{(s+2)(s+1)}$$

Compute terms:

$$a_2 = -\frac{a_0}{2!}, \qquad a_3 = -\frac{a_1}{3!}$$
$$a_4 = \frac{a_0}{4!}, \qquad a_5 = \frac{a_1}{5!}, \ldots$$

Series becomes:

$$y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

Final solution:

$$y(x) = a_0 \cos x + a_1 \sin x$$

Example 2.1.8. *Solve the following ODE by power series:*

$$y' = 2xy$$

Solution 2.1.7 (Solution to Example 2.1.8). *Substitute series:*

$$a_1 + \sum_{s=0}^{\infty} (s+2)a_{s+2}x^{s+1} = \sum_{s=0}^{\infty} 2a_s x^{s+1}$$

Match powers:

$$a_1 = 0, \quad (s+2)a_{s+2} = 2a_s \Rightarrow a_{s+2} = \frac{2}{s+2}a_s$$

Compute terms:

$$a_1 = 0, \quad a_2 = a_0, \quad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \quad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \ldots$$

Solution:

$$y = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) = a_0 e^{x^2}$$

Remark 2.1.2. *Note that we do not need power series method for these or similar ODEs? We used them just for explaining the idea of the method. What happens if we apply the method to an ODE not of the kind considered so far, even to an innocent-looking one such as $y'' + xy = 0$ ("Airy's equation")? We most likely end up with new special functions given by power series.*

2.2 Power series solution of general variable coefficient linear ODE

In the last section we saw that the power series method gives solutions of ODEs in the form of power series.

2.2.1 Homogeneous case

We consider solving a variable linear ODE of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{2.7}$$

If we divide Equation (2.7) by $P(x)$, we obtain

$$Ly := y'' + p(x)y' + q(x)y = 0, \tag{2.8}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. In order to calculate the higher derivatives of $y(x)$ to substitute into Taylors formula, we rewrite(2.8) as follows:

$$y'' = -p(x)y' - q(x)y$$

If $y(x_0)$ and $y'(x_0)$ are given, then $y''(x_0)$ can be obtained directly from the ODE. Higher derivatives of y can, in turn, be obtained by differentiating the ODE repeatedly. This process will be successful provided $p(x)$ and $q(x)$ are infinitely differentiable at $x = x_0$. **In this case, $p(x)$ and $q(x)$ are said to be analytic at** x_0 and have Taylor expansions of the form:

$$p(x) = p_0 + p_1(x - x_0) + \cdots = \sum_{k=0}^{\infty} p_k (x - x_0)^k$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots = \sum_{k=0}^{\infty} q_k (x - x_0)^k$$

Note

Note that in the power series method we can differentiate, add, and multiply power series, in a "suitable sense". For example: consider two power series

$$f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m \quad \text{and} \quad g(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m$$

Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of $x - x_0$ is

$$f(x) \cdot g(x) = \sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x - x_0)^m$$
$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \cdots$$

2.2.2 Ordinary points and singular points

Definition 2.2.1 (Ordinary points and singular points)**.** *The expansion point x_0 is said to be an ordinary point of (2.8) if $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$ are analytic at x_0. Otherwise, x_0 is a singular point.*

Theorem 2.2.1. *If x_0 is an ordinary point, it is possible to obtain power series expansions of the solution $y(x)$ of the form:*

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \tag{2.9}$$

then substitute it into equation (2.8) and solve for the unknown coefficients c_n in order to determine a solution. In addition, (2.9) converges at least on the open interval $(x_0 - R, x_0 + R)$, where R is the the distance from x_0 to the nearest singular point of (2.8).
More precisely, the radius of convergence of (2.9) is at least as large as the radius of convergence of each of the series expansions for $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$, i.e., up to the closest singularity to x_0.

Example 2.2.1 (Singular point).

1. *When $P, Q,$ and R are polynomials and $P(x_0) = 0$ while $Q(x_0) \neq 0$ or $R(x_0) \neq 0$, then x_0 is a singular point.*

2. *Another example is when:*

$$p(x) = \sqrt{x}, \quad q(x) = 2$$

then $x_0 = 0$ is a singular point because $p(x)$ is not differentiable at $x = 0$.

Observations

- If P, Q, and R are polynomials, then a point x_0 such that $P(x_0) \neq 0$ is an ordinary point.
- If $x_0 = 0$ is an ordinary point, then we assume:

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

The substitution into the ODE

$$Ly = 0$$

to get:

$$0=\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}+\left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{n=1}^{\infty} n c_n x^{n-1}\right)+\left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right).$$

Re-indexing the powers of x, this results in:

$$\sum_{m=0}^{\infty}\left\{(m+2)(m+1)c_{m+2}+\left(p_0(m+1)c_{m+1}+\cdots+p_m c_1\right)+\left(q_0 c_m+\cdots+q_m c_0\right)\right\}x^m=0$$

This yields a non-degenerate recursion formula for the c_m. At an ordinary point x_0, we can obtain two linearly independent solutions of the form (2.9).

Example 2.2.2. *Find the power series in x for the general solution of*

$$\left(1+2x^2\right)y''+6xy'+2y=0 \tag{2.10}$$

Solution 2.2.1 (Solution of example 2.2.2). *The functions $p(x)=\frac{6x}{1+2x^2}$ and $q(x)=\frac{2}{1+2x^2}$ are analytic at $x=0$. Hence, $x_0=0$ is an ordinary point of (2.10). Next, we rewrite the given equation as:*

$$\left(1+2x^2\right)y''+6xy'+2y=y''+2x^2y''+6xy'+2y=0$$

Since $x_0=0$ is an ordinary point of (2.10), we assume the general solution in the form of a power series as:

$$y=\sum_{n=0}^{\infty} a_n x^n$$

which will generate two linearly independent solutions. Now, differentiating y :

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1},\quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Substituting these into the differential equation:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+2x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+6x\sum_{n=1}^{\infty} n a_n x^{n-1}+2\sum_{n=0}^{\infty} a_n x^n=0$$

we can express the equation as:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}+\sum_{n=2}^{\infty} 2n(n-1)a_n x^n+\sum_{n=1}^{\infty} 6n a_n x^n+\sum_{n=0}^{\infty} 2a_n x^n=0$$

Let $m=n-2$ in the first sum then, $n=m+2$ and when $n=2, m=0$. We thus obtain after taking $n=m$ in the other sums:

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m+\sum_{m=2}^{\infty} 2m(m-1)a_m x^m+\sum_{m=1}^{\infty} 6m a_m x^m+\sum_{m=0}^{\infty} 2a_m x^m=0$$

Reindexing:

$$2a_2 + 6a_3x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{m=2}^{\infty} 2m(m-1)a_m x^m + 6a_1x + \sum_{m=2}^{\infty} 6ma_m x^m$$
$$+ 2a_0 + 2a_1x + \sum_{m=2}^{\infty} 2a_m x^m = 0$$

Next, we collect like terms:

$$(2a_2 + 2a_0) + (6a_3 + 8a_1)\,x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} + \left(2m^2+4m+2\right)a_m\right]x^m = 0$$

Equating the coefficients of powers of x, we get the recurrence relations for a_m :

$$a_2 = -a_0, \quad a_3 = -\frac{4}{3}a_1, \quad a_{m+2} = -\frac{2(m+1)^2}{(m+1)(m+2)}a_m = -\frac{2(m+1)}{(m+2)}a_m, \quad m = 2,3,4,\ldots. \tag{2.11}$$

Substituting $m = 2,3,4,\ldots$ into the recurrence relations (2.11):

$$a_4 = -\frac{3}{2}a_2 = \frac{3}{2}a_0, \quad a_5 = -\frac{8}{5}a_3 = \frac{32}{15}a_1, \quad a_6 = -\frac{5}{3}a_4 = -\frac{5}{2}a_0, \quad a_7 = -\frac{12}{7}a_5 = -\frac{128}{35}a_1, \ldots$$

Thus, the general solution is:

$$y = a_0\left(1 - x^2 + \frac{3}{2}x^4 - \frac{5}{2}x^6 + \cdots\right) + a_1\left(x - \frac{4}{3}x^3 + \frac{32}{15}x^5 - \frac{128}{35}x^7 + \cdots\right). \tag{2.12}$$

This is the power series in x for the general solution of equation (2.10). Now, let us compute the radius of convergence of this solution. We use the ratio test and the recurrence relation (2.11):

$$\lim_{m\to\infty}\left|\frac{a_{m+2}x^{m+2}}{a_m x^m}\right| = |x|^2 \lim_{m\to\infty}\left|-\frac{2(m+1)}{(m+2)}\right| = 2|x|^2$$

Therefore, the series converges if $2|x|^2 < 1$ i.e., $|x| < \frac{\sqrt{2}}{2}$. Hence, $R = \frac{\sqrt{2}}{2}$.
Another way to determine the radius of convergence is to find the nearest singular point of (2.10) (which could be a complex number) and compute the distance from $x_0 = 0$ to this nearest singular point. Since P is a polynomial, $P(x) = 1 + 2x^2$, the singular points are the roots of $P(x) = 0$, i.e.,

$$1 + 2x^2 = 0 \Longrightarrow x^2 = -\frac{1}{2} \Longrightarrow x = \pm i\frac{1}{\sqrt{2}}.$$

The singular points are therefore $x = \pm i\frac{\sqrt{2}}{2}$.
To compute the distance from $x_0 = 0$ to the nearest singular point, we calculate the modulus of $x = i\frac{\sqrt{2}}{2}$ (or of $x = -i\frac{\sqrt{2}}{2}$):

$$|x| = \left|i\frac{\sqrt{2}}{2}\right| = \frac{\sqrt{2}}{2}.$$

Hence according to Theorem 2.2.1 the series (2.12) converges at least on the open interval $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Example 2.2.3. *Consider the Airy equation, which arises in Quantum Mechanics:*

$$Ly = y'' - xy = 0$$

We observe that $x = 0$ is an ordinary point.

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} c_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

In the first sum, let $m+1 = n-2 \quad n = m+3 \quad n = 2 \Rightarrow m = -1$

$$c_2 2x^0 + \sum_{m=0}^{\infty} \left[c_{m+3}(m+3)(m+2) - c_m\right] x^{m+1} = 0$$

$$c_2 = 0, \quad c_{m+3} = \frac{c_m}{(m+3)(m+2)} \quad m = 0, 1, \ldots$$

(1) $c_0 \to c_3 \to c_6$.

$$c_3 = \frac{c_0}{3.2}, c_6 = \frac{c_3}{6.5} = \frac{c_0}{6.5.3.2}, c_9 = \frac{c_0}{9.8.6.5.3.2}$$

$$c_{3n} = \frac{c_0}{(3n)(3n-1)(3n-3)(3n-4)\ldots 9.8.6.5.3.2}$$

$$y_0(x) = 1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \cdots + \frac{x^{3n}}{(3n)(3n-1)\ldots 3.2} + \ldots$$

(2) $c_1 \to c_4 \to c_7 \to$.

$$c_4 = \frac{c_1}{4.3} \quad c_7 = \frac{c_1}{7.6.4.3} \quad c_{10} = \frac{c_1}{(10.9)(7.6)(4.3)}$$

$$c_{3n+1} = \frac{c_1}{(3n+1)(3n)(3n-2)(3n-3)\ldots(7.6)(4.3)}$$

$$y_1(x) = x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \cdots + \frac{x^{3n+1}}{(3n+1)(3n)\ldots 4.3}$$

$$y(x) = c_0 y_0(x) + c_1 y_1(x)$$

Radius of Convergence:

$$\lim_{m\to\infty} \left|\frac{c_{m+3}}{c_m} x^3\right| = \lim_{m\to\infty} \frac{|x|^3}{(m+3)(m+2)} = 0 < 1. \quad R = \infty$$

2.2.3 Solutions near other points

When looking for solutions near an ordinary point $x_0 \neq 0$, the calculations are generally simpler if we translate x_0 to the origin by the change of variable $t = x - x_0$.
The solution to the resulting differential equation can be obtained by the method of power series near $t = 0$.
The solution to the original equation is then obtained simply by performing the inverse translation.

Example 2.2.4. *To find the solution of the differential equation in powers of $(x-1)$ of the ODE:*

$$y'' - xy = 0 \tag{2.13}$$

Solution 2.2.2 (Solution to Example 2.2.4)**.** *The point $x_0 = 1$ is an ordinary point of (2.13). The solution of the equation is written in the form:*

$$y = \sum_{n=0}^{\infty} b_n (x-1)^n$$

Let

$$t = x - 1 \quad \textit{and} \quad dt = dx$$

Then the equation becomes (denoting derivatives with respect to t as y_t''):

$$y_t'' - (t+1)y = 0$$

and

$$y = \sum_{n=0}^{\infty} b_n t^n$$

Now,

$$y' = \sum_{n=1}^{\infty} n b_n t^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) b_n t^{n-2}$$

Substituting into the equation, we obtain:

$$\sum_{n=2}^{\infty} n(n-1) b_n t^{n-2} - (t+1) \sum_{n=0}^{\infty} b_n t^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) b_{n+2} t^n - \sum_{n=0}^{\infty} b_n t^n - \sum_{n=1}^{\infty} b_{n-1} t^n = 0$$

Then:

$$(2 \cdot 1\, b_2 - b_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) b_{n+2} - b_{n-1} - b_n \right] t^n = 0$$

Thus, we get the recurrence relation:

$$\begin{cases} b_2 = \frac{b_0}{2 \cdot 1} \\ b_{n+2} = \frac{1}{(n+2)(n+1)} \left(b_{n-1} + b_n \right), \quad n \geq 1 \end{cases}$$

For a few terms:

$$\begin{aligned} b_3 &= \frac{1}{3 \cdot 2}(b_0 + b_1) \\ b_4 &= \frac{1}{4 \cdot 3}(b_1 + b_2) = \frac{1}{4 \cdot 3 \cdot 2} b_0 + \frac{1}{4 \cdot 3} b_1 \\ b_5 &= \frac{1}{5 \cdot 4}(b_2 + b_3) = \frac{4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} b_0 + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} b_1 \end{aligned}$$

So,

$$\begin{aligned} y &= \sum_{n=0}^{\infty} b_n t^n \\ &= b_0 + b_1 t + b_2 t^2 + \left(\frac{1}{3\cdot 2} b_0 + \frac{1}{3\cdot 2} b_1\right) t^3 + \dots \\ &= b_0 \left(1 + \frac{1}{2} t^2 + \frac{1}{3\cdot 2} t^3 + \dots\right) + b_1 \left(t + \frac{1}{3\cdot 2} t^3 + \dots\right) \end{aligned}$$

Since $t = x - 1$, we have:

$$y = b_0 \left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{3\cdot 2}(x-1)^3 + \dots\right) + b_1 \left((x-1) + \frac{1}{3\cdot 2}(x-1)^3 + \dots\right)$$

Note

Example (2.2.4) can still be solved directly by assuming a power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n,$$

which gives you the same general solution as above.

2.2.4 Non-homogeneous case

Definition 2.2.2 (Ordinary point and singular point)**.** *Let*

$$y'' + p(x)y' + q(x)y = f(x) \tag{2.14}$$

We say that x_0 is an ordinary point of equation (2.14) if $p(x), q(x)$, and $f(x)$ are all analytic at x_0. Otherwise, x_0 is a singular point.

Theorem 2.2.2. *Let x_0 be an ordinary point of 2.14. Let R be the distance from x_0 to the nearest singular point of 2.14. Then every solution of 2.14 can be represented by a power series:*

$$y = \sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n, \tag{2.15}$$

that converges at least on the open interval $(x_0 - R, x_0 + R)$.
Furthermore, 2.15 will generate the two linearly independent solutions of the homogeneous part of 2.14 and a particular solution to the nonhomogeneous part of 2.14.

Example 2.2.5. *Find the power series in x for the general solution of*

$$y'' + xy' + y = \frac{1}{1-x} \tag{2.16}$$

Solution 2.2.3 (Solution to Example 2.2.5)**.** *The functions* $p(x) = x, q(x) = 1$ *and* $f(x) = \frac{1}{1-x}$ *are analytic at* $x = 0$*. The series for* $\frac{1}{1-x}$ *is given by:*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

which converges on $(-1, 1)$*. Since* $x = 0$ *is an ordinary point, Theorem 2.2.2 says we will generate both the homogeneous and the non-homogeneous solutions of* (2.16) *from:*

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these into (2.16)*, we get:*

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n$$

Reindexing the sums, we obtain:

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} x^n$$

Collecting like terms:

$$(2a_2 + a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n\right] x^n = 1 + \sum_{n=1}^{\infty} x^n$$

Equating coefficients, we find:

$$a_2 = \frac{1}{2}\left(1 - a_0\right), \quad a_{n+2} = -\frac{a_n}{n+2} + \frac{1}{(n+2)(n+1)}, \quad n = 1, 2, 3, \ldots$$

Substituting $n = 1, 2, 3, \ldots$*, we find:*

$$a_3 = -\frac{1}{3} a_1 + \frac{1}{6}, \quad a_4 = -\frac{1}{4} a_2 + \frac{1}{12} = \frac{1}{8} a_0 - \frac{1}{24}, \quad a_5 = -\frac{1}{5} a_3 + \frac{1}{20} = \frac{1}{15} a_1 + \frac{1}{60}, \ldots$$

Thus, the general solution is:

$$y = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 + \cdots\right) + a_1 \left(x - \frac{1}{3} x^3 + \frac{1}{15} x^5 + \cdots\right) + \left(\frac{1}{2} x^2 + \frac{1}{6} x^3 - \frac{1}{24} x^4 + \frac{1}{60} x^5 + \cdots\right)$$

This is the power series in x *for the general solution of Equation* (2.16)*, defined on* $(-1, 1)$.

Example 2.2.6. *Find a solution in series for the equation:*

$$y'' - xy' = e^{-x} \tag{2.17}$$

Solution 2.2.4 (Solution to Example 2.2.6). *We have: $p(x) = -x$ and $q(x) = 0$ and $f(x) = e^{-x}$, which are analytic at $x_0 = 0$. Therefore, $x_0 = 0$ is an ordinary point of (2.17).*
Let

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

and since

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

By substituting the series into equation (2.17), we obtain:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Thus,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Now,

$$2 \cdot 1 a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$$

We conclude that

$$\begin{cases} a_2 = \frac{1}{2} \\ a_{n+2} = \frac{(-1)^n}{(n+2)!} + \frac{n}{(n+2)(n+1)} a_n, \quad \forall n \geq 1 \end{cases}$$

For
$n = 1$,

$$a_3 = -\frac{1}{3!} + \frac{1}{3 \cdot 2} a_1, \quad a_4 = \frac{1}{4!} + \frac{2}{4 \cdot 3 \cdot 2} = \frac{2}{4!}$$

For $n = 3$,

$$a_5 = -\frac{1}{5!} + \frac{3}{5 \cdot 4} a_3 = -\frac{1}{5!} + \frac{3}{5 \cdot 4} \left[-\frac{1}{3!} + \frac{1}{3 \cdot 2} a_1 \right] = -\frac{4}{5!} + \frac{3}{5!} a_1$$

For $n = 4$,

$$a_6 = \frac{13}{6!}$$

For $n = 5$,

$$a_7 = \frac{-21}{7!} + \frac{15}{7!} a_1$$

Thus,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

By substituting the values of a_n into y, we obtain the general solution of (2.17):

$$y = a_0 \cdot 1 + a_1 \left[x + \frac{1}{3!} x^3 + \frac{3}{5!} x^5 + \frac{15}{7!} x^7 + \dots \right] + \left[\frac{1}{2} x^2 - \frac{1}{3!} x^3 + \frac{3}{4!} x^4 - \frac{4}{5!} x^5 + \dots \right].$$

You can now check using the recurrence formula that $R = \infty$.