Chapter 3

Introduction to partial differential equations

3.1 Introduction

This chapter introduces some basic partial differential equations (PDEs). A partial differential equation is a mathematical equation in which the unknown is a function of several variables and involves the partial derivatives of this function with respect to those variables. For example, one might want to determine the temperature at a specific point in space over time. In this case, the unknown function represents the temperature, and the PDE involves its partial derivatives with respect to time and spatial variables.

Partial differential equations appear in many models in physics, engineering, or biology, such as the propagation of heat or sound, fluid flow, electrodynamics, and the spread of epidemics. They are also used in weather forecasting models and climate models.

3.2 Classification of PDEs

In the previous chapters, we discussed linear ordinary differential equations (ODEs). We saw that these are equations that define functions of a single independent variable by establishing a relationship between the values of the function and its derivatives. Now, let us give an example of nonlinear ODE

Example 3.2.1 (Examples of nonlinear ODEs). *Some examples of nonlinear ODEs are given by:*

1. $(x^2 + y)y'(x) + 2xy(x) = y^2(x)$ *(first order);*
2. $y''(x) + e^{y(x)} = 0$ *(second order).*

PDEs involve multivariable functions $u(x, t)$, $u(x, y)$ that are determined by prescribing a relationship between the function value and its partial derivatives.

Definition 3.2.1 (Order of a PDE). *The order of a PDE is defined as the order of the highest partial derivative occurring in the equation.*
A PDE is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. Otherwise it is said to be non-linear

Note:

In the remainder of the chapter, we will occasionally use u_x to denote $\frac{\partial u}{\partial x}$, u_{xx} to denote $\frac{\partial^2 u}{\partial x^2}$, and u_{xy} to denote $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$.

Example 3.2.2

Example 3.2.2. *Let u be an unknown function dependent on x and y.*

1. *Linear first order PDE:*

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y), \tag{3.1}$$

where , a and b are not identically 0.

2. *Second order linear PDE:*

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{3.2}$$

where A, B, C, D, E, F, G are constants or functions of x, y and A, B, C are not identically 0.

If $G = 0$ the PDE is said to be homogeneous, if $G \neq 0$ the PDE is said to be non homogeneous.

Analogous to characterizing quadratic equations

$$AX^2 + BXY + CY^2 + DX + EY = k, \tag{3.3}$$

as hyperbolic, parabolic, or elliptic, determined by the sign of the discriminant: $\Delta = B^2 - 4AC$, we do the same for partial differential equations (PDEs). This brings us to the following classification.

Δ	Type of PDE	Quadric (Analogous)	Example of PDE	PDE Nature
$\Delta > 0$	Hyperbolic	$T^2 - c^2X^2 = k$	$u_{tt} = c^2 u_{xx}$	Wave equation
$\Delta = 0$	Parabolic	$T = X^2$	$u_t = u_{xx}$	Heat equation/ diffusion equation
$\Delta < 0$	Elliptic	$X^2 + Y^2 = k$	$u_{xx} + u_{yy} = f$	Laplace's equation if $f = 0$ Poisson equation if $f \neq 0$

All linear and second-order PDEs can be transformed into one of these types.

3.3 A one dimensional conservation law

Assume we are looking at the traffic flow at a length Δx of a highway as shown in Figure 3.1. We denote by $u(x,t)$ the density of cars at position x at time t.

$$[u] = \text{ number of cars /unit length.}$$

Let $q(x,t)$ be the flux of cars at position x at time t.

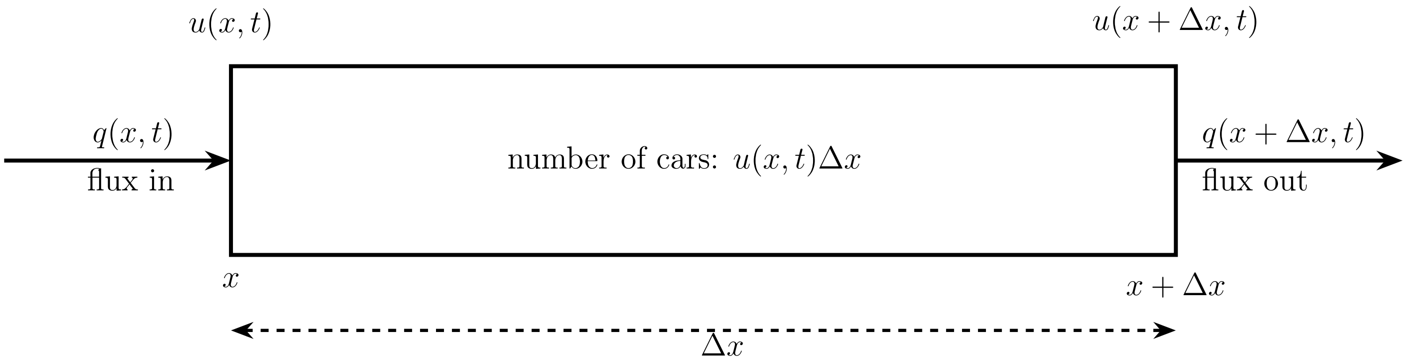

Figure 3.1: Traffic flow along the x axis with density $u(x,t)$ and flux $q(x,t)$ at x and time t.

$$[q] = \text{ number of cars / unit time.}$$

The conservation law tells us that the change in the number of cars over $[t, t+\Delta t]$ is equal to:

number of cars in - number of cars out.

That is

$$u(x, t+\Delta t)\Delta x - u(x,t)\Delta x = q(x,t)\Delta t - q(x+\Delta x, t)\Delta t. \tag{3.4}$$

Check dimensions:

$$\frac{\#\text{cars}}{L} \cdot L = \frac{\#\text{cars}}{T} \cdot T$$

Divide (3.4) by $\Delta t \cdot \Delta x$:

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{q(x,t) - q(x+\Delta x,t)}{\Delta x}$$

Now let $\Delta t \to 0$, $\Delta x \to 0$:

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x} \text{ or } \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{3.5}$$

Remark 3.3.1. *Equation (3.5) describes a PDE of conservation laws. In this equation, $u(x,t)$ and $q(x,t)$ are both unknowns. To solve this PDE we need to know how q is related to u.*
This information comes from the nature of the problem. For instance it can be: an equation of state (thermodynamics) or a constitutive relation (continuum mechanics).

3.3.1 Linear flux density relationship

Assume that the flux of cars q increases linearly with the density of cars u, i.e., $q = cu$, $c > 0$, then it follows that

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0} \tag{3.6}$$

Since the PDE (3.6) has constant coefficients and is a linear combination of time and spatial partial derivatives, we might expect to find a solution of the form of an exponential of a linear function of x

and t, since either derivative of such a function is in the form of a constant times the exponential. We therefore consider the trial solution of the form:

$$u(x,t) = \mathrm{e}^{\mathrm{i}kx+\sigma t} \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\mathrm{e}^{\mathrm{i}kx+\sigma t} = (\sigma + \mathrm{i}kc)\mathrm{e}^{\mathrm{i}kx+\sigma t}$$

which is a solution of (3.6) provided σ and k satisfy the following "dispersion relation"

$$\sigma = -\mathrm{i}kc$$

So, the solution of (3.6) is

$$\boxed{u(x,t) = e^{\mathrm{i}k(x-ct)}}$$

Remark 3.3.2. *It can be shown that any differentiable function f with the functional form $f(x-ct)$ is a solution to (3.6) : To see this, let $u(x,t) = f(x-ct)$, then $u_t = -cf'(x-ct)$ and $u_x = f'(x-ct)$ and $u_t + cu_x = -cf' + cf' = 0$. The PDE (3.6) with the solution $u(x,t) = f(x-ct)$ can be interpreted as a right moving wave using the Galilean transformation (see Figure 3.2). The wave propagates in time to the right:*

- *observer in blue, stationary, sees x;*
- *observer in red, moving with the wave, sees $x' = x - ct$.*

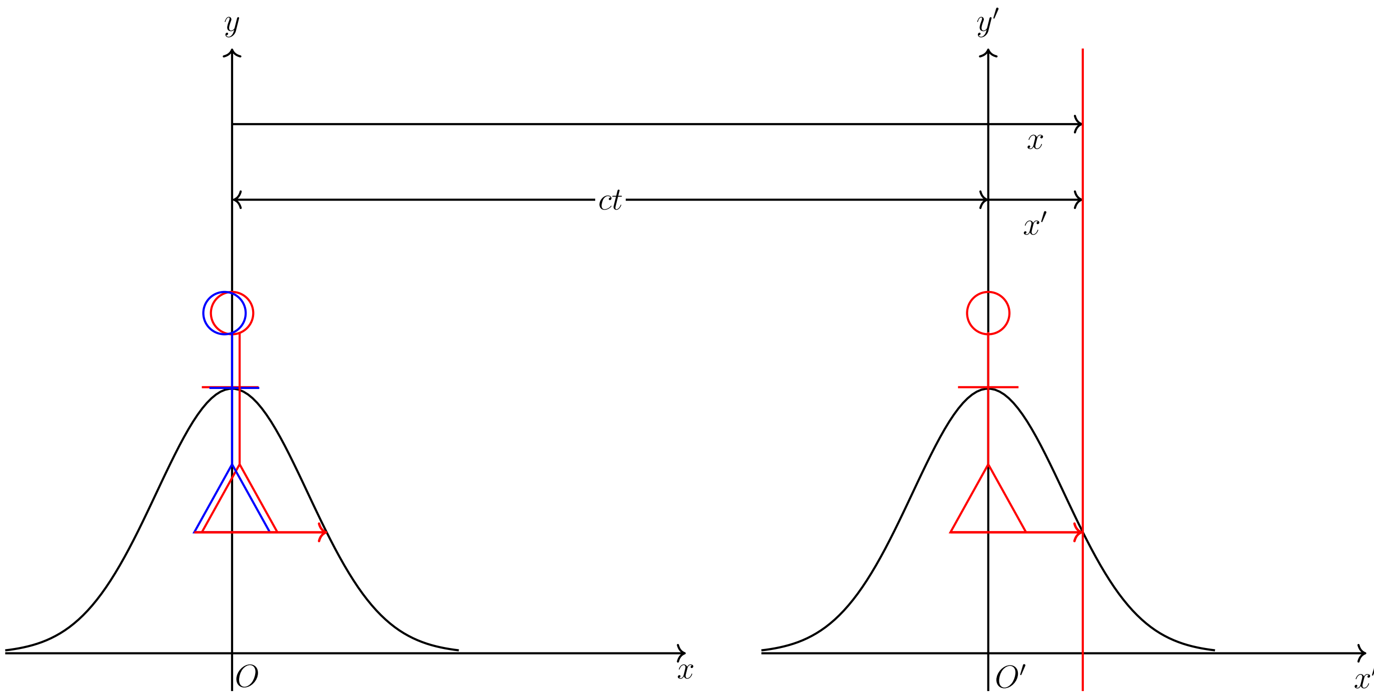

Figure 3.2: The Galilean transformation of coordinates from x to $x' = x - ct$.

Assume that the flux of cars q decreases linearly with the density of cars u, $q = -cu$, $c > 0$, i.e. the wave is moving to the left. Then it follows that

$$\boxed{\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0} \tag{3.8}$$

and the solution is $u(x,t) = f(x+ct)$.

3.3.2 The second order wave equation:

Here, we consider a wave moving in both directions. If we apply the left $\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}$ and right $\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}$ moving wave operators in succession, we obtain

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t) = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{3.9}$$

which is the second order wave equation that has both left and right moving wave solutions.

3.3.3 The convection-diffusion equation

Consider the traffic flowing down the highway as shown in Figure 3.1 and assume that the flux q increases linearly with the car density u. Now what happens if drivers slow down after seeing an increase in car density ahead of them? This situation can be represented by a flux function of the form;

$$\begin{gathered} q = cu - Du_x \\ \frac{\#\,\text{cars}}{T} = \frac{L}{T}\cdot\frac{\#\text{Cars}}{L} - [D]\frac{\#\text{Cars}}{L^2} \end{gathered} \tag{3.10}$$

so, D should have dimensions: $[D] = \frac{L^2}{T}$. Combining (3.6) and (3.10) we obtain the convection-diffusion equation

$$\underbrace{u_t + cu_x}_{\text{Convection}} = \underbrace{Du_{xx}}_{\text{Diffusion}} \tag{3.11}$$

So, now in addition to moving at speed c to the right, the wave diffuses too, with a diffusion coefficient D. Now, if you make a change of variable: $z = x - ct$, i.e., you move with the center of the wave, and find the PDE for $U(z)$:

$$U_t = DU_{zz} \tag{3.12}$$

This means the observer that travels with the wave only sees the diffusion.

Finding the dispersion relation for the Convection-diffusion equation

Consider a solution: $u(x,t) = e^{ikx+\sigma t}$. Substitute in:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = D\frac{\partial^2 u}{\partial x^2}$$

to obtain

$$(\sigma + ick)e^{ikx+\sigma t} = \left(-k^2\right)De^{ikx+\sigma t}$$

Hence,

$$\sigma = -\underbrace{ikC}_{\text{due to convection}} - \underbrace{k^2 D}_{\text{due to diffusion}} \quad \text{(the dispersion relation)}.$$

Therefore,

$$u(x,t) = e^{ikx-ikct-k^2Dt} = \underbrace{e^{ik(x-ct)}}_{\text{right moving wave}} \cdot \underbrace{e^{-k^2Dt}}_{\text{decay in time due to diffusion } D>0}$$

3.4 The heat/diffusion equation

Consider the heat conduction in a length Δx of a conducting bar as shown in Figure 3.3:

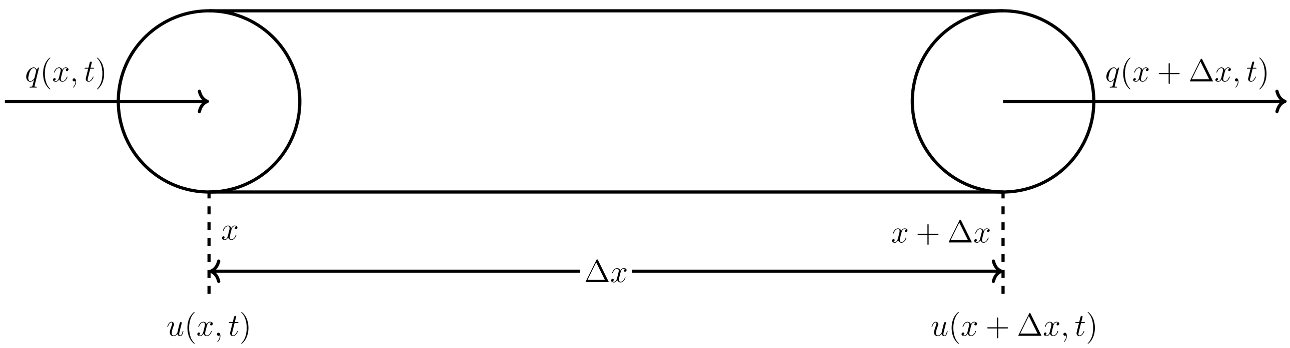

Figure 3.3: Heat conduction along the x axis.

- $u(x,t)$: The temperature at location x, time t, and has units degrees Kelvin, $[u] = K$;
- $q(x,t)$: The heat flux, or the flux of heat energy per unit area, $[q] = \frac{J}{m^2 \cdot S}$;
- C : The specific heat capacity. The amount of energy needed to increase the temperature of one kilogram of the material by one degree Kelvin: $[C] = \frac{J}{\text{kg}\cdot\text{ K}}$ (a material property);
- ρ : Density of the material, $[\rho] = \frac{\text{kg}}{\text{m}^3}$;
- A : The cross sectional area of the bar $[A] = m^2$.

Now, let us write down the conservation of energy:
The increase in the thermal energy of the bar with length Δx = thermal energy in thermal energy out. That is

$$C \cdot [u(x,t+\Delta t) - u(x,t)] \cdot \rho \cdot \Delta x \cdot A = [q(x,t) - q(x+\Delta x,t)]A \cdot \Delta t. \tag{3.13}$$

Check the dimensions in equation above:

$$\frac{J}{kg \cdot K} \cdot K \cdot \frac{kg}{m^3} \cdot m \cdot m^2 = \frac{J}{m^2 \cdot S} \cdot m^2 \cdot S$$
$$J = J$$

Divide (3.13) by $A \cdot \Delta x \cdot \Delta t$:

$$\rho C \frac{[u(x,t+\Delta t) - u(x,t)]}{\Delta t} = \frac{[q(x,t) - q(x+\Delta x,t)]}{\Delta x}$$

Let $\Delta t \to 0$ and $\Delta x \to 0$, we obtain $\rho C \frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x}$ i.e.,

$$\rho C \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad \text{(the energy conservation PDE)} \tag{3.14}$$

Now, we need to find a constitutive relation between u and q.

1. **Fourier's law:** Experimental evidence suggests that the flux of heat is proportional to the negative of the spatial gradient of the temperature. This means that heat always flows from higher temperature to lower temperature regions. In this case:

$$q = -k \frac{\partial u}{\partial x} \tag{3.15}$$

where k is the thermal conductivity having dimensions $[k] = \frac{J}{S \cdot m \cdot K} = \frac{W}{m \cdot K}$.

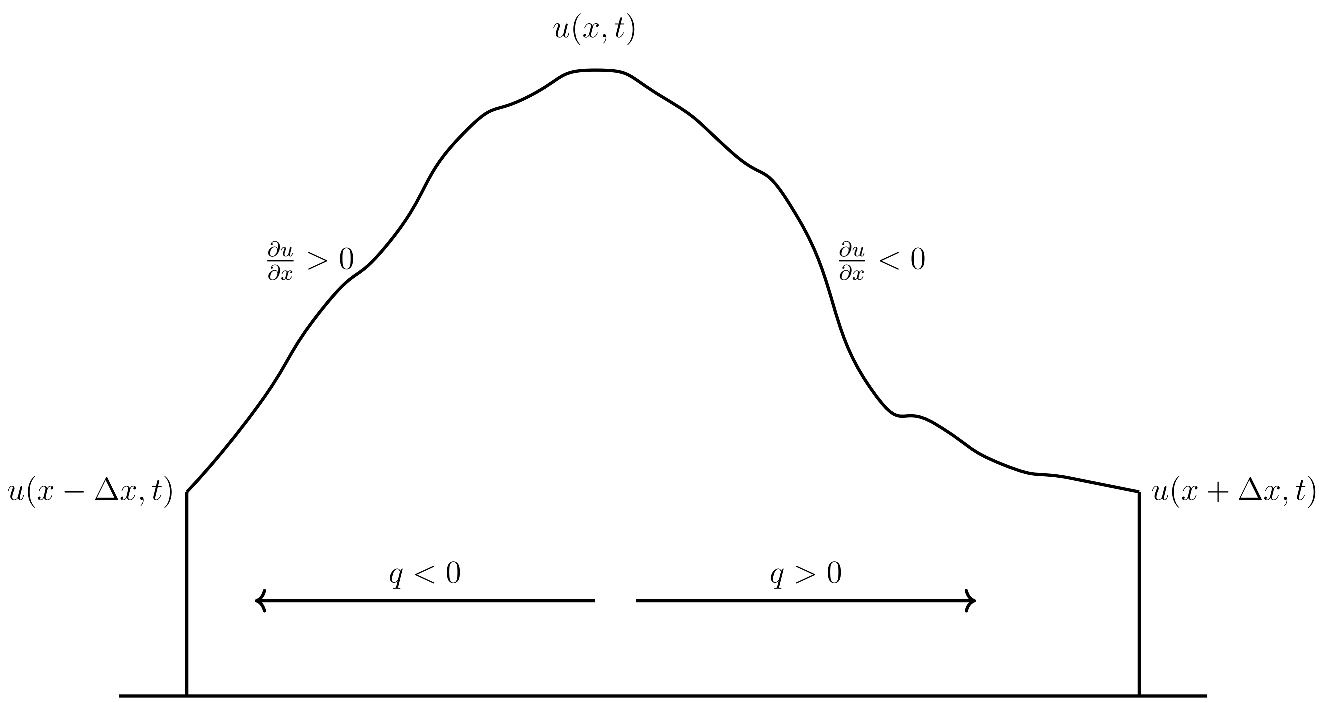

Figure 3.4: Fourier's Law of heat Conduction .

Substituting (3.15) into (3.14) and dividing by ρC we obtain the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{3.16}$$

where $\alpha^2 = \frac{k}{\rho C}$ is the diffusion coefficient, which has dimensions $[\alpha^2] = \frac{m^2}{S}$.

2. **Fick's law:** The heat flux is from regions of high Concentration of energy to regions of low concentration of energy.

$$q = -\alpha^2 \frac{\partial(\rho C u)}{\partial x} \tag{3.17}$$

Here, $\rho C u$ is the concentration of thermal energy, and has units: $[\rho C u] = \frac{\text{kg}}{\text{m}^3} \cdot \frac{\text{J}}{\text{kg} \cdot \text{ K}} \cdot K = \frac{J}{\text{m}^3}$. Substituting(3.17) into (3.14), we obtain

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{3.18}$$

where α^2 is the diffusion coefficient. A similar line of reasoning for the heat flow in a conduction plate leads to the two dimensional Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Another nice way of arriving at the diffusion equation is by random walk. See lecture 7 of Prof. Peirce's lectures.

3.5 The Wave Equation:

Consider an elastic rod having a density ρ and cross-sectional area A, and let $\sigma(x,t)$ be the pressure in the rod at x at time t and $u(x,t)$ the displacement of the rod from its equilibrium position as shown in Figure 3.5.

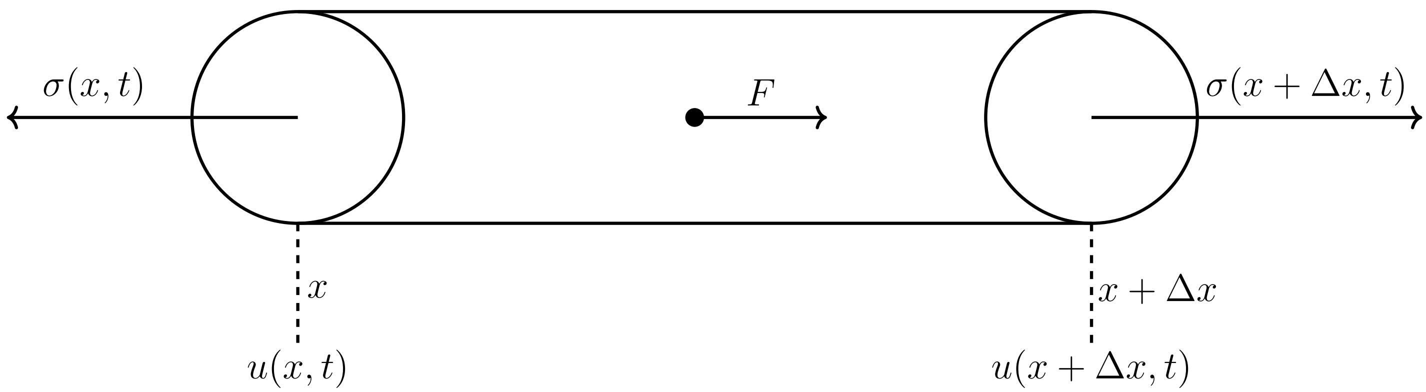

Figure 3.5: Displacement of a metal rod

- $u(x,t)$: displacement from equilibrium, $[u] = m$;
- $\sigma(x,t)$: the normal stress $[\sigma] = \frac{N}{m^2 \cdot S}$;
- ρ : density, $[\rho] = \frac{\text{kg}}{\text{m}^3}$;
- A : The cross sectional area of the bar $[A] = m^2$.

Now, let us write down Newton second law ($F = Ma$) :

$$\underbrace{[\sigma(x+\Delta x,t) - \sigma(x,t)] \cdot A}_{\text{net force}} = \underbrace{\rho \cdot \Delta x \cdot A}_{\text{mass}} \cdot \underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} \tag{3.19}$$

Divide (3.19) by $A \cdot \Delta x$:

$$\frac{\sigma(x+\Delta x,t) - \sigma(x,t)}{\Delta x} = \rho \frac{\partial^2 u}{\partial t^2}$$

Let $\Delta x \to 0$, we obtain

$$\frac{\partial \sigma}{\partial x} + \rho \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{(balance of linear momentum)} \tag{3.20}$$

Remark 3.5.1. *To solve the PDE (3.20), we need a constitutive law that gives a relation between σ and u. In order to have sufficient information to solve for the unknowns we need an additional equation, which is provided by a constitutive relation known as* **Hooke's Law** *(see Figure 3.6). Experimental data characterizes the "stiffness" of the material by the parameter E known as the Young's Modulus, which provides a linear relationship between the stress to which the bar is subjected and the relative displacement*

$$\frac{\Delta u}{\Delta x} = \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x} \approx \frac{\partial u}{\partial x} := \epsilon,$$

or strain ϵ.

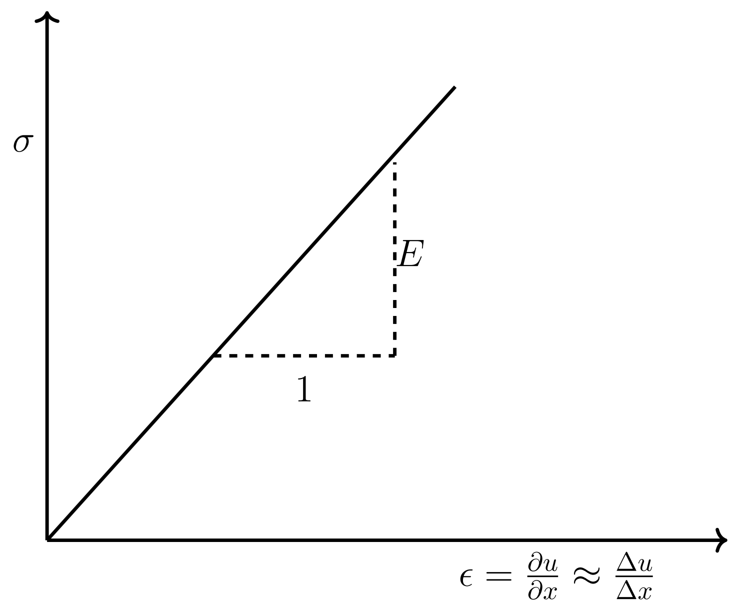

Figure 3.6: The stress on the bar σ is related to the strain ϵ by Hooke's Law Prof. Peirce's lectures.

Substituting the stress strain relationship $\sigma = E\frac{\partial u}{\partial x}$ into (3.20), we obtain the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where} \quad c = \sqrt{\frac{E}{\rho}}. \tag{3.21}$$

A general form of the solution of (3.21)to this equation is

$$u(x,t) = f(x - ct) + f(x + ct)$$

3.6 Laplace's equation: Flow in porous media

Consider the steady-state 2D flow in porous media aas shown in Figure 3.7.

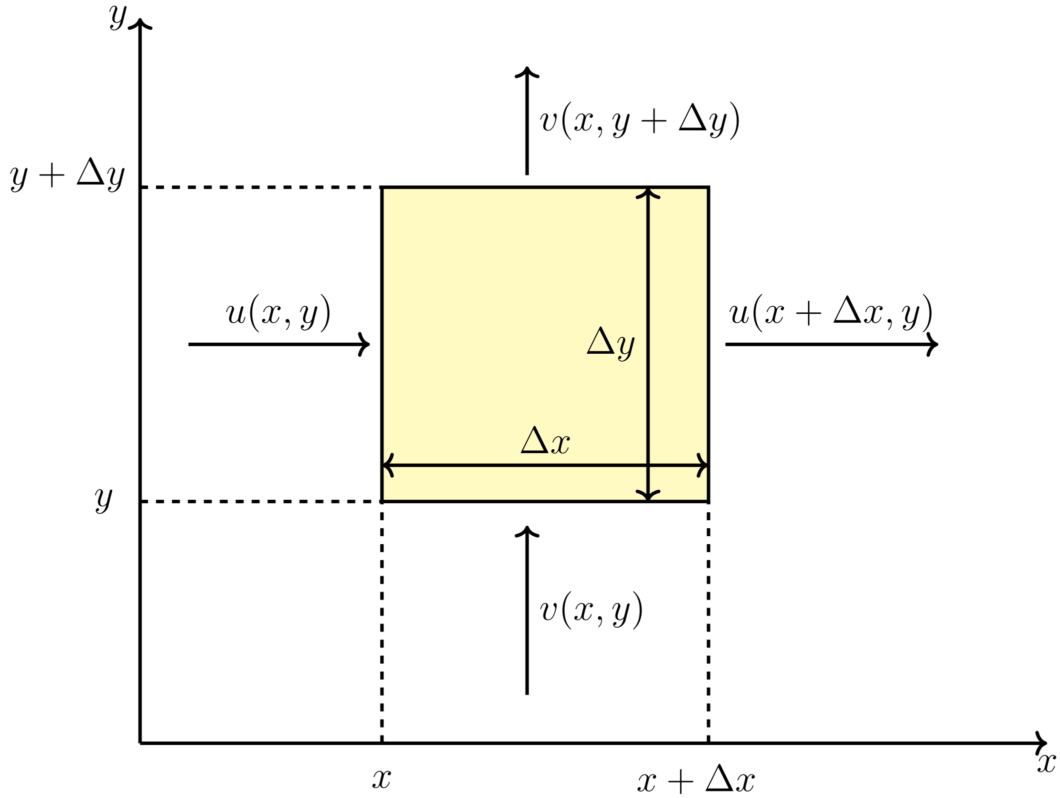

Figure 3.7: Steady state flow in a porous media.

- u : x component of velocity, $[u] = \text{m/S}$
- v : y component of velocity, $[V] = \text{m/S}$
- ρ : density, $[p] = \text{kg/m}^3$
- q : mass flux, $[q] = \text{kg}/S$

The Conservation of mass tells us that the sum of fluxes through all boundaries should be zero: We denote by l is a unit length in the y direction. The equation is:

$$\rho[u(x+\Delta x, y) - u(x, y)]\Delta y \cdot l + \rho[v(x, y+\Delta y) - v(x, y)]\Delta x \cdot l = 0. \tag{3.22}$$

Check the dimensions of each term:

$$\frac{\text{kg}}{m^3} \cdot \frac{m}{S} \cdot m \cdot m = \frac{\text{kg}}{S} \tag{3.23}$$

This ensures that the unit of mass flux is correct.
Next, divide Equation (3.22) by $\rho \Delta x \cdot \Delta y \cdot l$:

$$\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y} = 0$$

Let $\Delta x \to 0$ and $\Delta y \to 0$. This leads to the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.24}$$

Remark 3.6.1. *Now, we need to find a constitutive relation between u and v. For flow in porous media, you use Darcy's law as a constitutive law:*

$$u = -k\frac{\partial h}{\partial x}, \quad v = -k\frac{\partial h}{\partial y} \tag{3.25}$$

where:

- k *: hydraulic conductivity,* $[k] = \frac{m}{S}$
- h *: hydraulic head,* $[h] = \mathrm{m}$

So, Darcy's law states that the flow direction is from regions with higher hydraulic head to regions with lower hydraulic head.

Note

Darcy's law is sometimes stated as:

$$u = -\frac{x}{\mu}\frac{\partial P}{\partial x}, \quad \textit{where}$$

where:

- x *: permeability,* $[x] = \mathrm{m}^2$
- μ *: viscosity of fluid,* $[\mu] = \mathrm{Pa} \cdot S$
- P *: pore pressure,* $[P] = \mathrm{Pa}$

Substituting $u = -k\frac{\partial h}{\partial x}$ and $v = -k\frac{\partial h}{\partial y}$ into the continuity equation (3.24) gives the 2D Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{3.26}$$