Chapter 5

Fourier series and separation of variables

5.1 Introduction

In the previous chapter, we introduced the finite difference method as a numerical approach for solving a specific class of linear partial differential equations (PDEs). In this chapter, our goal is to solve these same PDEs analytically. We recall that a PDE is said to be linear if the dependent variable and its derivatives appear at most to the first power and in no functions.

To achieve this, we will employ the method of separation of variables, which involves transforming the PDE into a system of ordinary differential equations (ODEs).

General idea of the separation of variables

The idea of separation of variables is quite simple. Assume that you have a linear PDE along with some boundary and/or initial conditions. For clarity, let us assume that the PDE is linear and homogeneous, and that the boundary conditions are also linear and homogeneous. Assume also that the equation involves two variables: the first, referred to as time t, and the second, referred to as space x.

Methodology 5.1.1 (Idea of the separation of variables). *The method of separation of variables can be broken down into three steps:*

Step 1: *Find nonzero solutions of the PDE which have a product form*

$$u(x,t) = X(x)T(t).$$

Step 2: *Select from among the solutions found in* **Step 1** *those solutions which satisfy the BC. There will typically be an infinite sequence of these:*

$$u_n(x,t) = X_n(x)T_n(t), \quad n = 1, 2, \ldots$$

Step 3: *Observe that, because the PDE and BC are linear and homogeneous, any linear combination of solutions of these will again be a solution. Thus for any choice of coefficients* $b_1, b_2, \ldots$ *the linear combination*

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$$

will again be a solution of the PDE and BC (assuming the series converges). Choose the constants b_n so that $u(x,t)$ satisfies the initial condition.

5.2 Solution to the heat equation by separation of variables

Now, let us apply the previous method to the heat (diffusion) equation. We recall that the equation is given by:

$$u_t(x,t) = \alpha^2 u_{xx}(x,t), \quad 0 < x < L, \quad t > 0 \tag{5.1}$$

We consider the heat equation (5.1)subject to the following initial condition:

$$u(x,0) = f(x) \tag{5.2}$$

5.2.1 Dirichlet boundary conditions: Fourier sine series

Consider the heat conduction in an insulated rod whose endpoints are held at zero degrees for all time and within which the initial temperature is given by $f(x)$ as shown in Figure 5.1.

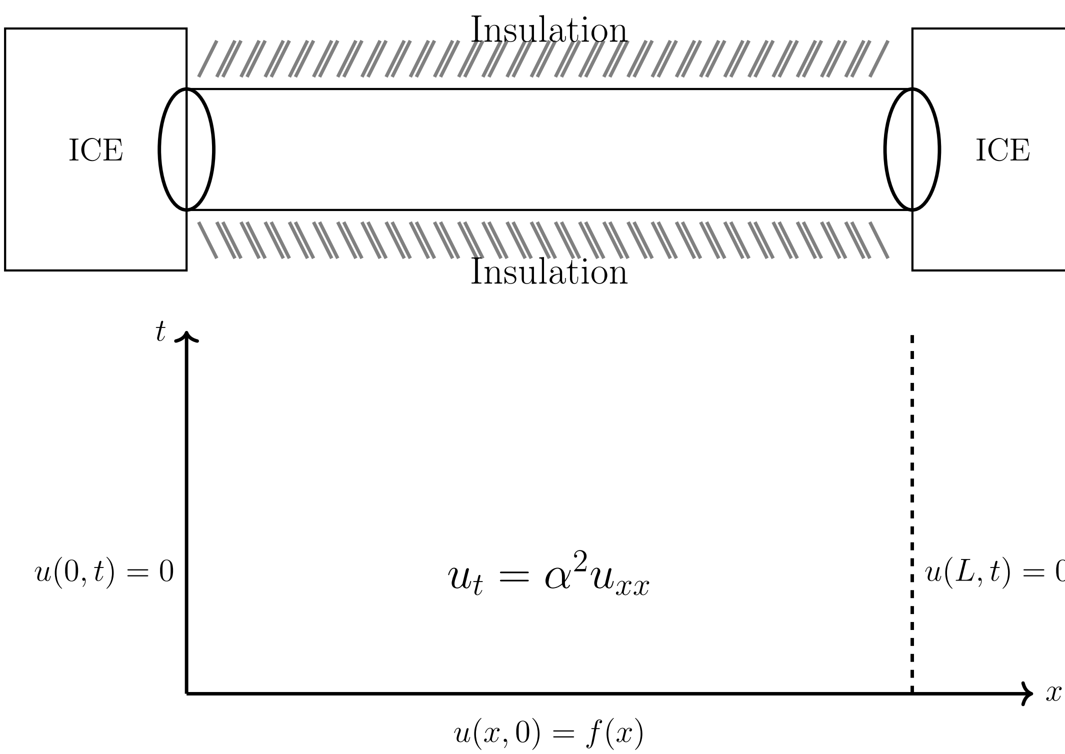


Figure 5.1: Consider a conducting bar with thermal conductivity α^2 that has an initial temperature distribution $u(x,0) = f(x)$ and whose endpoints are maintained at $0°C$, i.e. embedded in ice. See Prof. Peirce's lectures.

The system of equations is given by:

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, t > 0 \tag{5.3a}$$
$$\text{BC} : u(0,t) = 0, \quad u(L,t) = 0, \tag{5.3b}$$
$$\text{IC} : u(x,0) = f(x), \tag{5.3c}$$

We assume a solution of the form:

$$u(x,t) = X(x)T(t). \tag{5.4}$$

Differentiating both sides:

$$u_t = X(x) \cdot \dot{T}(t)$$
$$u_{xx} = X''(x) \cdot T(t)$$

Substituting these into the heat equation (5.3):

$$X(x) \cdot \dot{T}(t) = \alpha^2 X''(x) \cdot T(t). \tag{5.5}$$

Dividing by $\alpha^2 X(x)T(t)$:

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda \tag{5.6}$$

Since both sides depend on different variables, they must equal a constant, denoted as λ. This gives two ordinary differential equations (ODEs):

Time equation:

$$\dot{T}(t) = \lambda\alpha^2 T(t)$$

Solving this gives:

$$T(t) = Ce^{\lambda\alpha^2 t} \tag{5.7}$$

Space equation: an eigenvalue problem

$$X''(x) = \lambda X(x),\ X(0) = 0 = X(L) \tag{5.8}$$

An obvious solution of (5.8) is $X = 0$. This is a trivial solution. Can we find nontrivial solutions? The nature of $X(x)$ depends on λ.

Case 1; $\lambda > 0$: Let $\lambda = \mu^2$, then:

$$X'' - \mu^2 X = 0$$

The general solution is (we have chosen this form for simplifications, it is also fine to use the exponential form):

$$X(x) = A\sinh(\mu x) + B\cosh(\mu x) \tag{5.9}$$

Applying boundary conditions:

$$X(0) = 0 \Rightarrow B = 0, \quad X(L) = 0 \Rightarrow A \sinh(\mu L) = 0 \tag{5.10}$$

Since $\sinh(\mu L) \neq 0$, we must have $A = 0$, leading to the trivial solution.

Case 2: $\lambda = 0$: The equation simplifies to:

$$X''(x) = 0 \Rightarrow X(x) = Ax + B$$

Applying boundary conditions:

$$X(0) = 0 \Rightarrow B = 0, \quad X(L) = 0 \Rightarrow AL = 0 \Rightarrow A = 0 \tag{5.11}$$

Again, we get the trivial solution.

Case 3: $\lambda < 0$: Let $\lambda = -\mu^2$, then:

$$X'' + \mu^2 X = 0$$

The general solution is:

$$X(x) = A \sin(\mu x) + B \cos(\mu x). \tag{5.12}$$

Applying boundary conditions:

$$X(0) = 0 \Rightarrow B = 0, \quad X(L) = 0 \Rightarrow A \sin(\mu L) = 0$$

For a nontrivial solution ($A \neq 0$), we require:

$$\sin(\mu L) = 0 \Rightarrow \mu L = n\pi, \quad n = 1, 2, 3, \ldots \tag{5.13}$$

Thus,

$$\mu_n = \frac{n\pi}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2 \tag{5.14}$$

λ_n are eigenvalues. The corresponding eigenfunctions are:

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \tag{5.15}$$

So, the solution will have the form

$$u_n(x,t) = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), n = 1, 2, \ldots \tag{5.16}$$

Since the Equation (5.3) is linear, a linear combination of solutions is again a solution. Thus the most general solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right), \; b_n, \; n = 1, 2, \ldots.} \tag{5.17}$$

How can we find those coefficients?

Using the initial condition $u(x,0) = f(x)$:

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)} \tag{5.18}$$

This is the **Fourier sin series of** $f(x)$. We have the following question in mind:

Note!

Given a function $f(x)$ defined on $[0, L]$, do there exist constants $b_1, b_2, \ldots$ such that (5.18) holds? If the answer is "yes" then (5.17) is the solution to the heat equation problem (5.3).

We want to write the function $f(x)$ in terms of the sum of an infinite number of basis functions $\sin\left(\frac{n\pi x}{L}\right)$. This is similar to projecting a vector on a set of basis vectors. The sine function is periodic on the interval $[0, 2L]$ or $[-L, L]$:

$$\boxed{\begin{aligned} \sin\left(\frac{n\pi(x+2L)}{L}\right) &= \sin\left(\frac{n\pi x}{L} + 2n\pi\right) = \sin\left(\frac{n\pi x}{L}\right) \\ \sin\left(\frac{n\pi(x-L)}{L}\right) &= \sin\left(\frac{n\pi x}{L} - n\pi + 2n\pi\right) = \sin\left(\frac{n\pi(x+L)}{L}\right) \end{aligned}}$$

Now, we use an important property of trigonometric functions:

Theorem 5.2.1 (Orthogonality of trigonometric functions). *The trigonometric functions* $\sin\left(\frac{n\pi x}{L}\right)$ *and* $\cos\left(\frac{n\pi x}{L}\right)$ *(for* $n = 1, 2, 3, \ldots$*) are orthogonal over the interval* $[0, L]$ *in the following sense:*

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{L}{2}, & \text{if } n = m \end{cases} \tag{5.19a}$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{L}{2}, & \text{if } n = m \neq 0 \\ L, & \text{if } n = m = 0 \end{cases} \tag{5.19b}$$

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0, \quad \forall n, m \tag{5.19c}$$

The concept of orthogonality means that the inner product of two functions (or two vectors) is zero over an interval.

$$\langle f(x), g(x) \rangle = \int_0^L f(x) g(x) dx. \quad \textit{(Inner product of two functions)}$$

Proof of Theorem 5.2.1. To prove the first property (5.19a), we use the trigonometric identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Using these, we rewrite the product of two sine functions:

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

For $m \neq n$:

$$\begin{aligned}
\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx &= \int_0^L \frac{1}{2}\left[\cos\left(\frac{\pi x(n-m)}{L}\right) - \cos\left(\frac{\pi x(n+m)}{L}\right)\right]dx \\
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{\pi x(n-m)}{L}\right)\Bigg|_0^L - \frac{L}{(n+m)\pi}\sin\left(\frac{\pi x(n+m)}{L}\right)\Bigg|_0^L\right] \\
&= 0
\end{aligned}$$

For $m = n$, we use:

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$$

Thus:

$$\begin{aligned}
\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx &= \int_0^L \frac{1}{2}\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)dx \\
&= \frac{1}{2}\left[x\big|_0^L - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Bigg|_0^L\right] \\
&= \frac{L}{2}
\end{aligned}$$

The case of (5.19b) and (5.19c) follow with similar arguments. □

Finding Fourier coefficients b_n

Going back to our Fourier series representation:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

To find b_n, multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate over $[0, L]$:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

So:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = 0 + 0 + \cdots + b_m \int_0^L \sin^2\left(\frac{m\pi x}{L}\right)dx + 0 + 0 + \ldots$$

all terms in the sum, except the m^{th} term are zero. Hence,

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = b_m \cdot \frac{L}{2}$$

Thus, the Fourier sine coefficients are:

$$b_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx, \quad m = 1, 2, 3, \ldots$$

Finally, the most general solution to (5.3) is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{5.20}$$

Example 5.2.1 (Fourier sine expansion)**.** *Let us solve* (5.3) *with* $f(x) = x, 0 < x < 1, L = 1$. *We have using the integration by part formula,*

$$b_n = 2\int_0^1 x \sin(n\pi x) dx = 2\frac{(-1)^{n+1}}{n\pi}$$

Hence,

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\alpha^2 (n\pi)^2 t} \sin(n\pi x)$$

We can use the latter expression to compute some series: for example, if $t = 0$ *and* $x = \frac{1}{2}$, *then we have*

$$u(1/2, 0) = f(1/2) = 1/2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi/2)$$

k	n	$\sin(n\pi/2)$
0	1	1
	2	0
1	3	-1
	4	0
2	5	1

Therefore,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}$$

Example 5.2.2. *We want to solve the following PDE problem:*

$$u_t = 0.003 u_{xx}, \quad 0 < x < 1, t > 0$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = 50x(1-x) \quad \text{for } 0 < x < 1$$

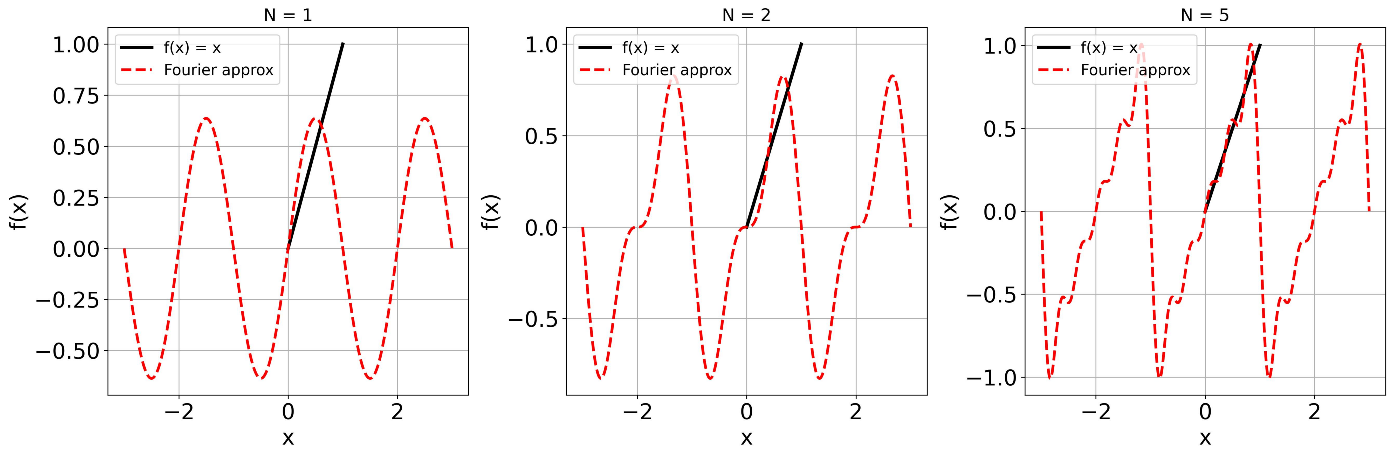


Figure 5.2: Representation of the terms of the Fourier series of $f(x) = x$.

As previously, we need to write $f(x) = 50x(1-x)$ for $0 < x < 1$ as a sine series. That is, $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$, where

$$b_n = 2\int_0^1 50x(1-x)\sin(n\pi x)dx = \frac{200}{\pi^3 n^3} - \frac{200(-1)^n}{\pi^3 n^3} = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{400}{\pi^3 n^3} & \text{if } n \text{ odd} \end{cases}$$

Hence the solution $u(x,t)$, is given by:

$$u(x,t) = \frac{400}{\pi^3}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} \sin((2k+1)\pi x)e^{-(2k+1)^2\pi^2 0.003t}$$

5.2.2 Neumann boundary conditions: Fourier cosine series

Consider the heat conduction in an insulated rod whose endpoints are insulated and within which the initial temperature is given by $f(x)$ as shown in Figure 5.3

The system of equations is given by:

$$PDE : u_t = \alpha^2 u_{xx}, \quad 0 < x < L,\ t > 0 \tag{5.21a}$$
$$\text{BC}: \quad u_x(0,t) = 0, \quad u_x(L,t) = 0, \tag{5.21b}$$
$$\text{IC}: \quad u(x,0) = f(x), \tag{5.21c}$$

We assume a solution of the form:

$$u(x,t) = X(x)T(t) \tag{5.22}$$

Differentiating both sides:

$$u_t = X(x)\cdot \dot{T}(t)$$
$$u_{xx} = X''(x)\cdot T(t)$$

Substituting these into the heat Equation (5.21):

$$X(x)\cdot \dot{T}(t) = \alpha^2 X''(x)\cdot T(t) \tag{5.23}$$

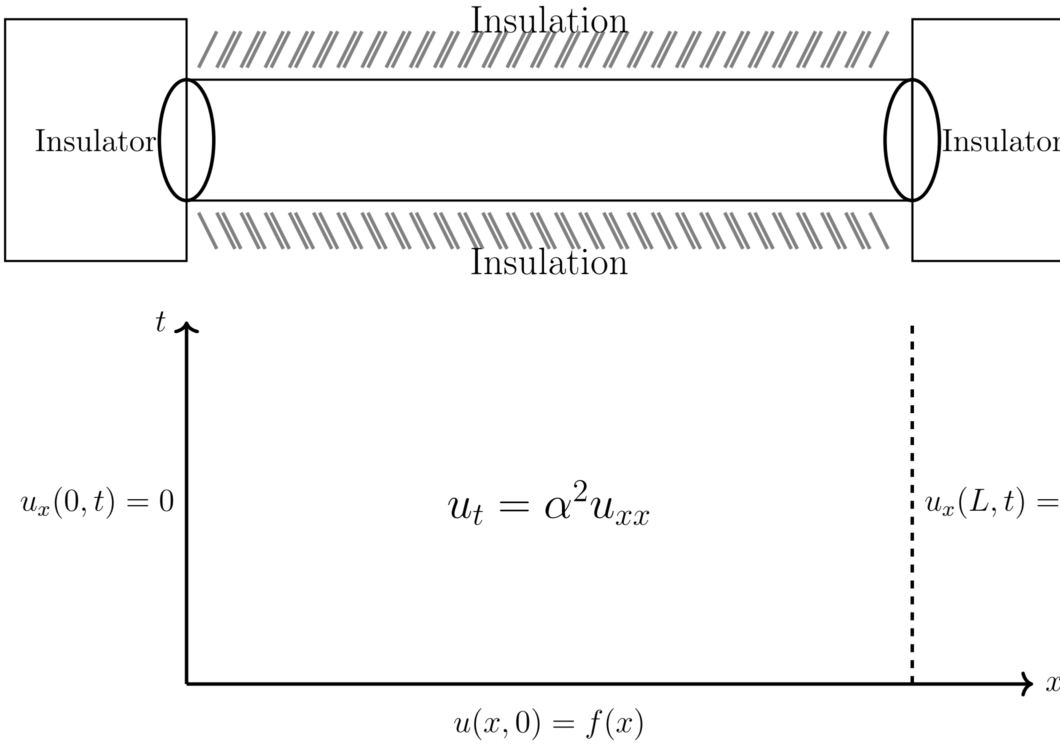


Figure 5.3: Consider a conducting bar with thermal conductivity α^{2^x} that has an initial temperature distribution $u(x,0) = f(x)$ and whose endpoints are insulated, Prof. Peirce's lectures.

Dividing by $\alpha^2 X(x)T(t)$:

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda \tag{5.24}$$

Since both sides depend on different variables, they must equal a constant, denoted as λ. This gives two ordinary differential equations (ODEs):

Time equation:

$$\dot{T}(t) = \lambda \alpha^2 T(t)$$

Solving this gives:

$$T(t) = Ce^{\lambda \alpha^2 t} \tag{5.25}$$

Space equation: an eigenvalue problem

$$X''(x) = \lambda X(x),\ X'(0) = 0 = X'(L) \tag{5.26}$$

The nature of $X(x)$ depends on λ.

Case 1: $\lambda > 0$**:** Let $\lambda = \mu^2$, then:

$$X'' - \mu^2 X = 0$$

The general solution is (we have chosen this form for simplifications, it is also fine to use the exponential form):

$$X(x) = A\sinh(\mu x) + B\cosh(\mu x). \tag{5.27}$$

We note that

$$X'(x) = A\mu\cosh(\mu x) + B\mu\sinh(\mu x)$$

Applying boundary conditions:

$$X'(0) = 0 \Rightarrow A = 0, \quad X'(L) = 0 \Rightarrow B\mu\sinh(\mu L) = 0 \tag{5.28}$$

Since $\sinh(\mu L) \neq 0$, we must have $B = 0$, leading to the trivial solution.

Case 2: $\lambda = 0$: The equation simplifies to:

$$X''(x) = 0 \Rightarrow X(x) = Bx + A$$

Applying boundary conditions:

$$X'(0) = 0 \Rightarrow B = 0, \quad X'(L) = 0 \Rightarrow B = 0 \tag{5.29}$$

So, $X(x) = A$ is a non-trivial solution. The eigenvalue $\lambda_0 = 0$ and the corresponding eigenfunction is $X_0(x) = 1$.

Case 3: $\lambda < 0$: Let $\lambda = -\mu^2$, then:

$$X'' + \mu^2 X = 0$$

The general solution is:

$$X(x) = A\sin(\mu x) + B\cos(\mu x) \tag{5.30}$$

We have

$$X'(x) = \mu A\cos(\mu x) - \mu B\sin(\mu x) \tag{5.31}$$

Applying boundary conditions:

$$X'(0) = 0 \Rightarrow A = 0, \quad X'(L) = 0 \Rightarrow -\mu B\sin(\mu L) = 0$$

For a nontrivial solution ($B \neq 0$), we require:

$$\sin(\mu L) = 0 \Rightarrow \mu L = n\pi, \quad n = 1, 2, 3, \ldots \tag{5.32}$$

Thus,

$$\mu_n = \frac{n\pi}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2 \tag{5.33}$$

λ_n are eigenvalues. The corresponding eigenfunctions are:

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots \tag{5.34}$$

So, the solution will have the form

$$\boxed{u_n(x,t) = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right), n = 1, 2, \ldots} \tag{5.35}$$

We also had another eigenvalue/function from $\lambda = 0, u_0(x,t) = A_0 \cdot e^{0\cdot t} = A_0$. Since the equation (5.21) is linear, a linear combination of solutions is again a solution. Thus the most general solution is of the form:

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right),} \quad A_n, n = 0, 1, 2, \ldots \text{ are constants} \tag{5.36}$$

Using the initial condition $u(x,0) = f(x)$:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \tag{5.37}$$

As previously, we use the inner product $\langle\cdot,\cdot\rangle$ to project $f(x)$ onto the basis functions in the series: We multiply both sides of (5.37) by $X_0(x) = 1$ and integrate over $[0, L]$:

$$\int_0^L f(x)dx = A_0\int_0^L 1dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

i.e

$$\int_0^L f(x)dx = A_0 \cdot x|_0^L + \sum_{n=1}^{\infty} A_n \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = A_0 L + 0$$

Hence,

$$A_0 = \frac{1}{L}\int_0^L f(x)dx$$

Now, if we multiply both sides of (5.37) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate over $[0, L]$:

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_0\int_0^L \cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

i.e.

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = A_0 \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L + A_m \cdot \frac{L}{2}$$

So,

$$A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

Thus, the Fourier cosine coefficients are:

$$\boxed{A_0 = \frac{1}{L}\int_0^L f(x)dx, \quad A_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx, \quad m = 1, 2, 3, \ldots} \tag{5.38}$$

The final solution is:

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dx + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right)\cos\left(\frac{n\pi}{L}x\right)e^{-\alpha^2\frac{n^2\pi^2}{L^2}t} \tag{5.39}$$

We observe that as $t \to \infty$ it follows that $u(x,t) \to A_0 = \frac{1}{L}\int_0^L f(x)dx$, which is just the average value of the initial heat $f(x)$ distributed in the bar. This is consistent with physical intuition. It is sometimes convenient to re-define the Fourier coefficients as follows:

$$a_0 = 2A_0$$
$$a_k = A_k,\ k = 1, 2, \ldots$$

so that the a_k can be rewritten on a unified form

$$a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx \quad k = 0, 1, 2, \ldots$$

In terms of the new coefficients a_k defined, the Fourier expansion for the initial condition function $f(x)$ is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \tag{5.40}$$

while the solution of the heat equation (5.23) is of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \tag{5.41}$$

Example 5.2.3 (Fourier cosine expansion)**.** *Determine the Fourier coefficients a_k for the function*

$$f(x) = x, \quad 0 < x < 1 = L$$

and use the resulting Fourier cosine expansion to prove the identity

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

We have,

$$a_0 = 2\int_0^1 x dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1 \textit{ and}$$
$$a_n = 2\int_0^1 x\cos(n\pi x)dx = 2\frac{(-1)^n - 1}{n^2\pi^2} = \begin{cases} -\frac{4}{n^2\pi^2}, & n \textit{ odd} \\ 0, & n \textit{ even}\end{cases}$$

Therefore,

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)\pi x) \tag{5.42}$$

To obtain the required identity we set $x = 1$ in and rearrange terms. We can also deduce that the solution of (5.23) with the initial condition $u(x,0) = x$ is given by

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos((2k+1)\pi x)\mathrm{e}^{-\alpha^2((2k+1)\pi)^2 t} \tag{5.43}$$

The partial sums are shown in Figure 5.4

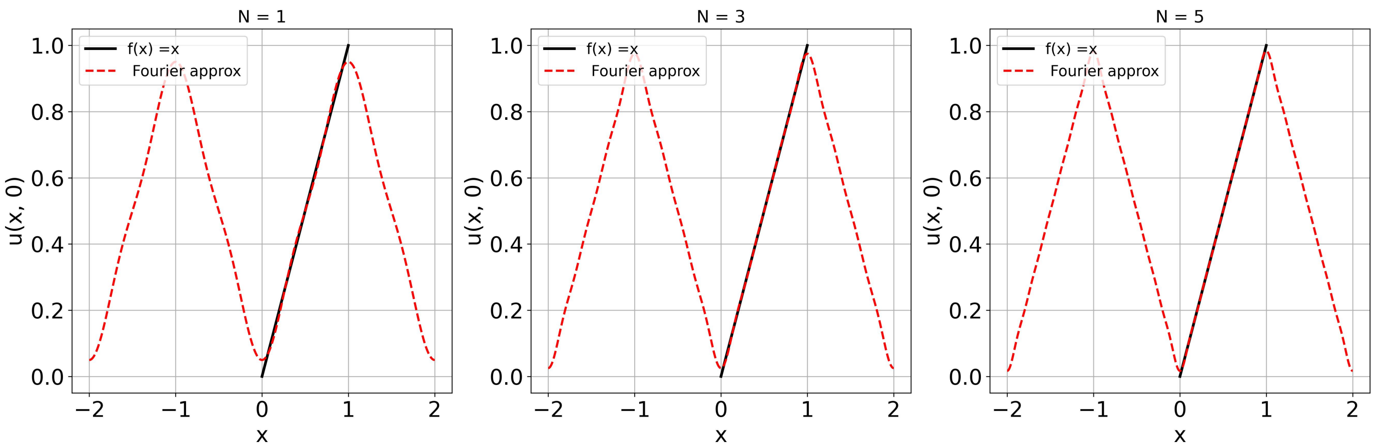


Figure 5.4: Partial sums of the Fourier Cosine Series of the function $f(x) = x$.

Example 5.2.4. *Let us solve now the following PDE problem*

$$\begin{aligned} &u_t = 0.003u_{xx}, \quad 0 < x < 1, t > 0 \\ &u_x(0,t) = u_x(1,t) = 0, \\ &u(x,0) = 50x(1-x) \quad \textit{for } 0 < x < 1. \end{aligned}$$

We must find the cosine series of $u(x,0)$. For $0 < x < 1$ we have

$$50x(1-x) = \frac{25}{3} - \frac{50}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos(2k\pi x)$$

Hence, the solution to the PDE problem, is given by:

$$u(x,t) = \frac{25}{3} - \frac{50}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos(2k\pi x)e^{-k^2\pi^2 0.012t}$$

5.2.3 Heat equation on a circular ring - Full range Fourier series

Consider a thin circular wire in which there is no radial temperature dependence as shown in Figiure 5.5, i.e., $u(r,\theta) = u(\theta)$ so that $\frac{\partial u}{\partial r} = 0$. In this case the polar Laplacian reduces to

$$\begin{aligned} \Delta u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\ &= \frac{\partial^2 u}{\partial (r\theta)^2} \end{aligned}$$

and if we let $x = r\theta$ then $\frac{\partial^2 u}{\partial (r\theta)^2} = u_{xx}$. In this case the heat distribution in the ring is determined by the following initial value problem with periodic boundary conditions:

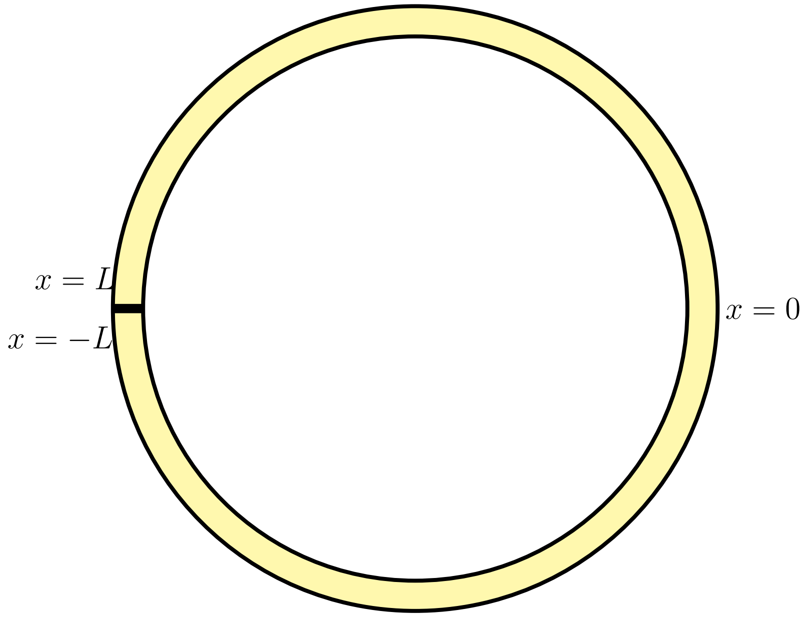


Figure 5.5: Consider a thin conducting ring with thermal conductivity α^2 that has a given initial temperature distribution.

$$\text{PDE}: u_t = \alpha^2 u_{xx}, \quad 0 < x < 2L, t > 0 \tag{5.44a}$$
$$\text{BC}: u(-L,t) = u(L,t), \quad u_x(-L,t) = u_x(L,t), \tag{5.44b}$$
$$\text{IC}: u(x,0) = f(x), \tag{5.44c}$$

We assume a solution of the form:

$$u(x,t) = X(x)T(t) \tag{5.45}$$

Differentiating both sides:

$$u_t = X(x) \cdot \dot{T}(t)$$
$$u_{xx} = X''(x) \cdot T(t)$$

Substituting these into the heat equation (5.44):

$$X(x) \cdot \dot{T}(t) = \alpha^2 X''(x) \cdot T(t) \tag{5.46}$$

Dividing by $\alpha^2 X(x)T(t)$:

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda. \tag{5.47}$$

Since both sides depend on different variables, they must equal a constant, denoted as λ. This gives two ordinary differential equations (ODEs):

Time equation:

$$\dot{T}(t) = \lambda \alpha^2 T(t)$$

Solving this gives:

$$T(t) = Ce^{\lambda\alpha^2 t}. \tag{5.48}$$

Space equation: an eigenvalue problem

$$X''(x) = \lambda X(x),\ X(-L) = X(L),\ X'(-L) = X'(L) \tag{5.49}$$

The nature of $X(x)$ depends on λ.

Case 1: $\lambda > 0$: Let $\lambda = \mu^2$, then:

$$X'' - \mu^2 X = 0$$

The general solution is (we have chosen this form for simplifications, it is also fine to use the exponential form):

$$X(x) = A\cosh(\mu x) + B\sinh(\mu x) \tag{5.50}$$

We note that

$$X'(x) = A\mu\sinh(\mu x) + B\mu\cosh(\mu x)$$

Applying boundary conditions:

$$X(-L) = X(L) \Rightarrow 2B\sinh(\mu L) = 0 \Rightarrow B = 0 \tag{5.51}$$

and

$$X'(-L) = X'(L) \Rightarrow 2A\mu\sinh(\mu L) = 0 \Rightarrow A = 0 \tag{5.52}$$

leading to the trivial solution.

Case 2: $\lambda = 0$ The equation simplifies to:

$$X''(x) = 0 \Rightarrow X(x) = Ax + B$$

We also have

$$X'(x) = A$$

Applying boundary conditions:

$$X(-L) = X(L) \Rightarrow 2AL = 0 \Rightarrow A = 0$$

and

$$X'(-L) = A = 0 = X'(L)$$

So, $X(x) = B$ is a non trivial solution. For this case, the eigenvalue $\lambda_0 = 0$ and the corresponding eigenfunction is $X_0 = 1$.

Case 3: $\lambda < 0$: Let $\lambda = -\mu^2$, then:

$$X'' + \mu^2 X = 0$$

The general solution is:

$$X(x) = A\cos(\mu x) + B\sin(\mu x) \tag{5.53}$$

We have

$$X'(x) = -\mu A\sin(\mu x) + \mu B\cos(\mu x) \tag{5.54}$$

Applying boundary conditions:

$$X(-L) = X(L) \Rightarrow 2B\sin(\mu L) = 0$$

and

$$X'(-L) = X'(L) \Rightarrow 2A\mu\sin(\mu L) = 0$$

For a nontrivial solution $(A, B, \mu \neq 0)$, we require:

$$\sin(\mu L) = 0 \Rightarrow \mu L = n\pi, \quad n = 1, 2, 3, \ldots \tag{5.55}$$

Thus,

$$\mu_n = \frac{n\pi}{L}, \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2 \tag{5.56}$$

λ_n are eigenvalues. The corresponding eigenfunctions are:

$$X_n(x) \in \left\{\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}, \quad n = 1, 2, 3, \ldots \tag{5.57}$$

So, the solution will have the form

$$u_n(x,t) = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right], n = 1, 2, \ldots \tag{5.58}$$

We also had another eigenvalue/function from $\lambda = 0$, $u_0(x,t) = A_0 \cdot e^{0\cdot t} = A_0$.
Since the equation (5.44) is linear, a linear combination of solutions is again a solution. Thus the most general solution is of the form:

$$\boxed{u(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right]} \tag{5.59}$$

for some coefficients $A_0, A_n, B_n, n = 1, 2, \ldots$.
Using the initial condition $u(x,0) = f(x)$:

$$f(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right] \tag{5.60}$$

As previously, we use the inner product $\langle\cdot,\cdot\rangle$ to project $f(x)$ onto the basis functions in the series. We state the following important result:

Theorem 5.2.2 (Orthogonality of trigonometric functions). *The trigonometric functions* $\sin\left(\frac{n\pi x}{L}\right)$ *and* $\cos\left(\frac{n\pi x}{L}\right)$ *(for* $n = 1, 2, 3, \ldots$ *) are orthogonal over the interval* $[-L, L]$ *in the following sense:*

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ L, & \text{if } n = m \end{cases} \tag{5.61a}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & \text{if } n \neq m \\ L, & \text{if } n = m \neq 0 \\ 2L, & \text{if } n = m = 0 \end{cases} \tag{5.61b}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0, \quad \text{for all } n, m \tag{5.61c}$$

The concept of orthogonality means that the inner product of two functions (or two vectors) is zero over an interval.

$$\langle f(x), g(x) \rangle = \int_{-L}^{L} f(x) g(x) dx. \quad \text{(Inner product of two functions)}$$

We multiply both sides of (5.60) by $X_0(x) = 1$ and integrate over $[-L, L]$:

$$\int_{-L}^{L} f(x)dx = A_0 \int_{-L}^{L} 1dx + \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx + \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx$$

i.e

$$\int_{-L}^{L} f(x)dx = A_0 \cdot x|_{-L}^{L} + \sum_{n=1}^{\infty} A_n \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\bigg|_{-L}^{L} - \sum_{n=1}^{\infty} B_n \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\bigg|_{-L}^{L} = 2A_0 L + 0 + 0$$

Hence,

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)dx \tag{5.62}$$

Now, if we multiply both sides of (5.60) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate over $[-L, L]$:

$$\int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx = A_0 \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) dx + \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$
$$+ \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

i.e. (using (5.61a) and (5.61c))

$$\int_{-L}^{L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx = A_0 \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\bigg|_{-L}^{L} + A_m L + 0$$

So,

$$A_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx \tag{5.63}$$

Now, if we multiply both sides of (5.60) by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate over $[-L, L]$:

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = A_0\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$
$$+\sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

i.e. (using (5.61b) and (5.61c))

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = -\ A_0\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L} + 0 + B_m L$$

So,

$$B_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx \tag{5.64}$$

So, in summary, the full Fourier series of $f(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0, 1, 2, 3, \ldots \tag{5.65}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx, \quad n = 1, 2, 3, \ldots \tag{5.66}$$

The final solution is:

$$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]} \tag{5.67}$$

We observe that as $t \to \infty$ it follows that $u(x,t) \to \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)dx$, which is just the average value of the initial heat $f(x)$.

Example 5.2.5 (Full Fourier expansion)**.** *Let us solve the equation (5.44) with the initial condition $u(x, 0) = f(x)$, where*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x \le \pi \end{cases}$$

In this case, $L = \pi$. We need to write the full Fourier expansion of $u(x, 0)$. We have,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x \ \mathrm{d}x = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x \\ &= \frac{1}{\pi}\left\{ x\frac{\sin(nx)}{n}\bigg|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\mathrm{d}x \right\} \\ &= \frac{1}{\pi}\left\{ \frac{\pi\sin(n\pi)}{n} + \frac{1}{n^2}\cos(nx)\bigg|_0^{\pi} \right\} \\ &= \frac{1}{\pi n^2}\left[(-1)^n - 1\right] \end{aligned} \tag{5.68}$$

For even indices, $a_{2m} = 0$ for $m = 0, 1, \ldots$. For odd indices:

$$a_{2m+1} = -\frac{2}{\pi(2m+1)^2}, \quad m = 0, 1, 2, \ldots$$

Now, we compute b_n

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\mathrm{d}x \\ &= \frac{1}{\pi}\left\{ -\ x\frac{\cos(nx)}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\mathrm{d}x \right\} \\ &= \frac{1}{\pi}\left\{ -\pi\frac{\cos(n\pi)}{n} + \frac{1}{n^2}\sin(nx)\bigg|_0^{\pi} \right\} \end{aligned}$$

Therefore, the Fourier series representation of $f(x)$ is given by

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) \\ &= \frac{\pi}{4} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\cos[(2m+1)x]}{(2m+1)^2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n} \end{aligned} \tag{5.69}$$

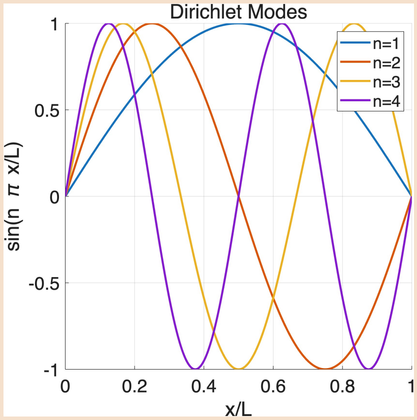
Finally, the solution takes the form:

$$\boxed{u(x,t) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{e^{-\alpha^2(2m+1)^2 t}\cos[(2m+1)x]}{(2m+1)^2} + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{e^{-\alpha^2 m^2 t}\sin(mx)}{m}} \tag{5.70}$$

Important summary!

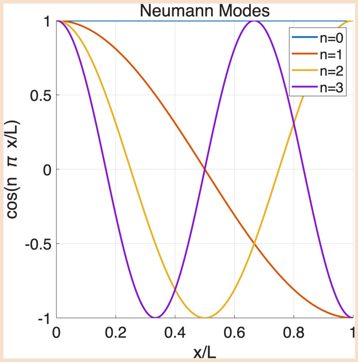
I: The Dirichlet Problem (Ice on Both Sides)

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(L) \end{cases} \implies \begin{cases} \lambda_n = \frac{n\pi}{L}, & n = 1, 2, \ldots \\ X_n(x) = \sin \frac{n\pi x}{L} \end{cases}$$



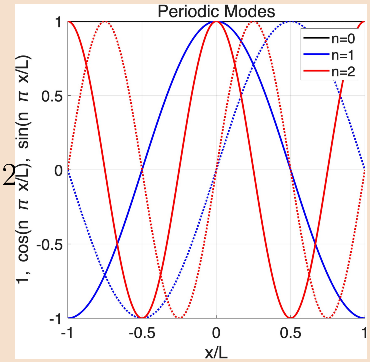
II: The Neumann Problem (Insulation on Both Sides)

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X^0(L) \end{cases} \implies \begin{cases} \lambda_n = \frac{n\pi}{L}, & n = 0, 1, 2, \ldots \\ X_n(x) = \cos \frac{n\pi x}{L} \end{cases}$$



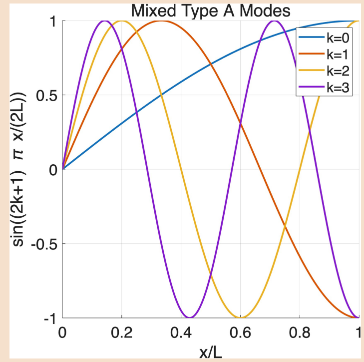
III: The Periodic Boundary Value Problem (The Closed Ring)

$$\begin{cases} X'' + \lambda^2 X = 0 \\ X(-L) = X(L) \\ X'(-L) = X'(L) \end{cases} \implies \begin{cases} \lambda_n = \frac{n\pi}{L}, & n = 0, 1, 2 \\ X_n(x) \in \{1, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}\} \end{cases}$$



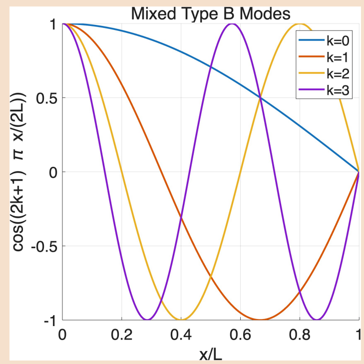
IV: Mixed Boundary Value Problem A (Ice Left and Insulation Right)

$$\begin{cases} X^{00} + \lambda^2 X = 0 \\ X(0) = 0 = X^0(L) \end{cases} \implies \begin{cases} \lambda_k = \frac{(2k+1)\pi}{2L}, \\ X_n(x) = \sin\frac{(2k+1)\pi x}{2L} \end{cases} \quad k = 0, 1, 2, \ldots$$



V: Mixed Boundary Value Problem B (Insulation Left and Ice Right)

$$\begin{cases} X^{00} + \lambda^2 X = 0 \\ X^0(0) = 0 = X(L) \end{cases} \implies \begin{cases} \lambda_k = \frac{(2k+1)\pi}{2L}, \\ X_n(x) = \cos\frac{(2k+1)\pi x}{2L} \end{cases} \quad k = 0, 1, 2, \ldots$$



5.3 Fourier Series

In the previous sections, we solved the heat equation under various boundary conditions, including periodic, Dirichlet, and Neumann conditions. In each case, a key step in the solution was expressing the initial condition $u(x,0)$ as a sum of sine and/or cosine functions. This type of expansion known as a **Fourier series**. The fundamental idea behind Fourier series is that any "reasonable" function can be represented as an infinite sum of trigonometric terms. In the following section, we explore Fourier series in more detail, developing the formalism that allows us to decompose functions into their fundamental frequency components.

We consider the expansion of the function $f(x)$ of the form

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = S(x), \tag{5.71}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad \frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \text{ average value of } f \tag{5.72}$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{5.73}$$

We recall that a function $f(x)$ defined for all x is periodic with period T if $f(x+T) = f(x)$ for all x.

We observe that $\cos\left(\frac{n\pi}{L}(x+T)\right) = \cos\left(\frac{n\pi x}{L}\right)$ provided $\frac{n\pi T}{L} = 2\pi,\ T = \frac{2L}{n}$ and similarly $\sin\left(\frac{n\pi}{L}(x+2L)\right) = \sin\left(\frac{n\pi x}{L}\right)$. Thus each of the terms of the Fourier Series $S(x)$ on the RHS of (5.71) is a periodic function with a maximal period $2L$ (a constant function is periodic with any period). As a result the function $S(x)$ is also periodic.

But the question is: **How does this relate to $f(x)$ which may not be periodic?**

The function $S(x)$ represented by the series is known as the periodic extension of f on $[-L, L]$.

Definition 5.3.1 (Periodic extension). *If f is defined on the interval $[a, b]$ then the periodic extension f_{per} of f, which has period $T = b - a$, is defined simply by "repeating" f in all the intervals $[a + nT, b + nT]$ for $n = 0, \pm 1, \ldots$, so that for all x,*

$$f_{per}(x) = f(x - nT) \quad \text{whenever } a + nT < x \leq b + nT, \quad n = 0, \pm 1, \pm 2, \ldots \tag{5.74}$$

In Figure 5.6, we show a picture for $a = -1, b = 1$, and functions $f(x) = x^2$ and $f(x) = x$.

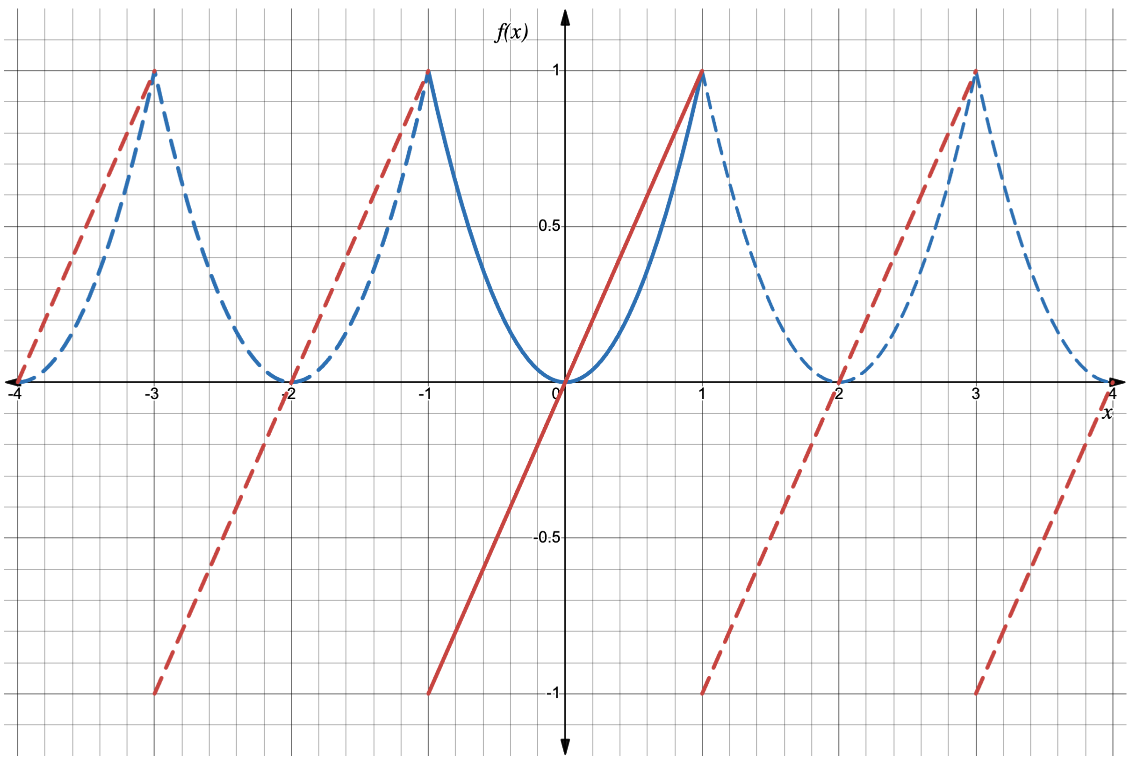


Figure 5.6: Periodic extension $f_{\text{per}}(x)$ of the function $f(x) = x^2$ (blue curve) and the function $f(x) = x$ (red curve) with $a = -1$ and $b = 1$.

Note that f_{per} may be discontinuous at a, b, etc., even if f is continuous. A related fact is that in defining f_{per} we have taken $f_{\text{per}}(a) = f(b)$ and not $f_{\text{per}}(a) = f(a)$; some choice must be made but this has no effect in practice.

Note: t can be useful to shift the interval of integration

Since the periodic extension f_{per} is periodic with period $2L$ (as are the basis functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{n\pi x}{L}\right)$), the interval $[-L, L]$ over which the integration is carried out may be replaced by any other interval of the same length: that is for any X,

$$a_0 = \frac{1}{2L}\int_X^{X+2L} f_{\text{per}}(x)dx, \quad a_n = \frac{1}{L}\int_X^{X+2L} f_{\text{per}}(x)\cos\frac{n\pi x}{L}dx, \quad n \geq 1$$
$$b_n = \frac{1}{L}\int_X^{X+2L} f_{\text{per}}(x)\sin\frac{n\pi x}{L}dx, \quad n \geq 1$$

Example 5.3.1. *Consider the example given in Example 5.2.5. Then, we have*

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \leq x \leq \pi \end{cases}$$

On $[\pi, 3\pi]$,

$$f_{per}(x) = \begin{cases} 0 & \pi < x < 2\pi \\ x - 2\pi & 2\pi \leq x \leq 3\pi \end{cases}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_\pi^{3\pi} f_{per}(x)\cos(nx)dx, \quad \textit{change of variables: } t = x - 2\pi \quad dx = dt, x = t + 2\pi \\ &= \frac{1}{\pi}\int_{2\pi}^{3\pi}(x-2\pi)\cos(nx)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} t\cos(nt)dt \quad \textit{since} \quad \cos n(t+2\pi) = \cos nt \end{aligned}$$

5.3.1 Half range Fourier Series: even and odd functions

We consider the Fourier Expansions for Even and Odd functions, which give rise to cosine and sine half range Fourier Expansions. If we are only given values of a function $f(x)$ over half of the range $[0, L]$, we can define two different extensions of f to the full range $[-L, L]$, which yield distinct Fourier Expansions. The even extension gives rise to a half range cosine series, while the odd extension gives rise to a half range sine series.

We first recall the elementary definitions of even, odd, and periodic functions.

Definition 5.3.2 (Even and odd functions). *A function $f(x)$ is said to be even if it is defined for all x (or possibly in some interval symmetric about $x = 0$, that is, of the form ($-L, L$) or $[-L, L]$) and satisfies $f(x) = f(-x)$; it is odd if it is similarly defined and satisfies $f(-x) = -f(x)$. We will frequently use the observation that if $f(x)$ is defined for $-L \leq x \leq L$ then,*

$$\int_{-L}^{L} f(x)dx = \begin{cases} 0, & \textit{if } f \textit{ is odd} \\ 2\int_0^L f(x)dx, & \textit{if } f \textit{ is even} \end{cases} \tag{5.75}$$

This formula is easily be derived by writing $\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_0^L f(x)dx$ and making the change of variable $y = -x$ in the first integral.

Note

Let $E(x)$ represent an even function and $O(x)$ an odd function. Then,
(a) If $f(x) = E(x) \cdot O(x)$ then $f(-x) = E(-x)O(-x) = -E(x)O(x) = -f(x) \Rightarrow f$ is odd.

(b) $E_1(x) \cdot E_2(x) \to$ even.
(c) $O_1(x) \cdot O_2(x) \to$ even.
(d) Any function can be expressed as a sum of an even part and an odd part:

$$f(x) = \frac{1}{2}\underbrace{[f(x) + f(-x)]}_{\text{even part}} + \frac{1}{2}\underbrace{[f(x) - f(-x)]}_{\text{odd part}}$$

Check: Let $E(x) = \frac{1}{2}[f(x) + f(-x)]$. Then $E(-x) = \frac{1}{2}[f(-x) + f(x)] = E(x)$ (even.)
Similarly let $O(x) = \frac{1}{2}[f(x) - f(-x)]$. Then, $O(-x) = \frac{1}{2}[f(-x) - f(x)] = -O(x)$ (odd.)

Important!: Consequences of the Even/Odd property for Fourier Series

One may use (5.75) to considerably simplify the formulas (5.72)-(5.73) when f is even or odd.

- If f is even then $f(x)\cos(n\pi x/L)$ is even and $f(x)\sin(n\pi x/L)$ is odd, so that from (5.75),

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = 0, \quad n \geq 1 \tag{5.76}$$

- If f is odd one has $f(x)\cos(n\pi x/L)$ is odd and $f(x)\sin(n\pi x/L)$ is even

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad n \geq 1 \tag{5.77}$$

- Since any function can be written as the sum of an even and an odd part, we can interpret the cosine and sine series as corresponding to even and odd functions:

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

where the first term represents the even part of $f(x)$ and the second term represents the odd part. Thus, the Fourier series expansion can be expressed as:

$$f(x) = \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\right\} + \left\{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right\},$$

where the cosine terms correspond to the even part and the sine terms correspond to the odd part of the function. In addition,

$$a_n = \frac{2}{L}\int_0^L \frac{1}{2}[f(x) + f(-x)]\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$b_n = \frac{2}{L}\int_0^L \frac{1}{2}[f(x) - f(-x)]\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Let us emphasize that in (5.76)-(5.77) we are considering the Fourier series of a function defined on the interval $[-L, L]$.

5.3.2 Half-range expansions

If we are given a function $f(x)$ on an interval $[0, L]$ and we want to represent f by a Fourier series, we have two choices: a **cosine series** or a **sine series**.

Cosine series:

If $f(x)$ is extended as an even function on $[-L, L]$, it can be represented by a Fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where the Fourier coefficients are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

We note that the even periodic extension is obtained by simply computing the Fourier series representation for the even function

$$f_e(x) \equiv \begin{cases} f(x), & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases}$$

Since $f_e(x)$ is an even function on a symmetric interval $[-L, L]$, we expect that the resulting Fourier series will not contain sine terms.
We can simplify this by noting that the integrand is even and the interval of integration can be replaced by $[0, L]$. On this interval $f_e(x) = f(x)$. So, we have the Cosine Series representation of $f(x)$ for $x \in [0, L]$ given as above.

Sine series:

If $f(x)$ is extended as an odd function on $[-L, L]$, it can be represented by a Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Similarly as the case of the cosine series, given $f(x)$ defined on $[0, L]$, the odd periodic extension is obtained by simply computing the Fourier series representation for the odd function

$$f_o(x) \equiv \begin{cases} f(x), & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases}$$

The resulting series expansion leads to defining the Sine Series representation of $f(x)$ for $x \in [0, L]$ as described above.

Example 5.3.2 (Half-Range Expansion of $f(x) = x$)**.** *Expand $f(x) = x$ on $0 < x < 2$ in a half-range (a) sine series, (b) cosine series.*

(a) Sine Series *($L = 2$).*

$$b_n = \frac{2}{L}\int_0^L x\sin\Big(\frac{n\pi}{L}x\Big)\,dx = \int_0^2 x\sin\Big(\tfrac{n\pi}{2}x\Big)\,dx$$
$$= -\left.\frac{x\cos\big(\frac{n\pi}{2}x\big)}{\frac{n\pi}{2}}\right|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\Big(\tfrac{n\pi}{2}x\Big)\,dx$$
$$= -\frac{4}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\sin\big(\tfrac{n\pi}{2}x\big)\Big|_0^2 = -\frac{4}{n\pi}(-1)^n.$$

Hence

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\Big(\tfrac{n\pi}{2}x\Big),$$

and at $x = 1$:

$$1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\Big(\tfrac{n\pi}{2}\Big) \implies \frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots.$$

(b) Cosine Series *($L = 2$).*

$$a_0 = \frac{2}{2}\int_0^2 x\,dx = \left.\frac{x^2}{2}\right|_0^2 = 2,$$
$$a_n = \int_0^2 x\cos\Big(\tfrac{n\pi}{2}x\Big)\,dx = \frac{2}{n\pi}x\sin\Big(\tfrac{n\pi}{2}x\Big)\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\Big(\tfrac{n\pi}{2}x\Big)\,dx + \Big(\tfrac{2}{n\pi}\Big)^2\cos\Big(\tfrac{n\pi}{2}x\Big)\Big|_0^2$$
$$= \frac{4}{n^2\pi^2}\big[\cos(n\pi) - 1\big].$$

Thus

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\Big(\tfrac{n\pi}{2}x\Big) = 1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos\big(\frac{(2k+1)\pi}{2}x\big)}{(2k+1)^2}.$$

At $x = 2$:

$$2 = 1 + \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \implies \frac{\pi^2}{8} = 1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \cdots.$$

Example 5.3.3 (Cosine Series of $\sin x$ on $0 \le x \le \pi$). *Find the Fourier cosine series of $f(x) = \sin x$ on $[0, \pi]$.*

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{4}{\pi}, \quad a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = 0.$$

For $n \ge 2$, use $2\sin x\cos(nx) = \sin((n+1)x) - \sin((n-1)x)$:

$$a_n = \frac{2}{\pi}\int_0^{\pi}\sin x\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi}[\sin((n+1)x) - \sin((n-1)x)]\,dx$$
$$= \frac{1}{\pi}\left[\tfrac{\cos((n-1)x)}{n-1} - \tfrac{\cos((n+1)x)}{n+1}\right]_0^{\pi} = \frac{2\big((-1)^{n-1} - 1\big)}{\pi(n^2-1)}.$$

Hence

$$\sin x = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}-1}{n^2-1}\cos(nx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\cos(2jx)}{(2j)^2-1}.$$

Example 5.3.4 (Periodic Extension of $f(x)=x$)**.** *Assume $f(x)=x$ on $0<x<2$ with period 2 so that $L=1$. Compute its full Fourier series.*

$$a_0 = \frac{1}{L}\int_{-L}^{L} x\,dx = \int_{-1}^{1} x\,dx = 0? \quad \textit{(equivalently compute } \int_0^2 x\,dx = 2 \textit{ for even part.)}$$

For $n \geq 1$:

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-1}^{1} x\cos(n\pi x)\,dx = \int_0^2 x\cos(n\pi x)\,dx \\ &= \left[\tfrac{x\sin(n\pi x)}{n\pi}\right]_0^2 - \frac{1}{n\pi}\int_0^2 \sin(n\pi x)\,dx = 0, \\ b_n &= \frac{1}{L}\int_{-1}^{1} x\sin(n\pi x)\,dx = \int_0^2 x\sin(n\pi x)\,dx \\ &= \left[-\tfrac{x\cos(n\pi x)}{n\pi}\right]_0^2 + \frac{1}{n\pi}\int_0^2 \cos(n\pi x)\,dx = -\frac{2}{n\pi}. \end{aligned}$$

Therefore

$$f(x) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n}.$$

Why??

In the previous example, if we compute $f(0)$ using the original definition of f, we obtain 0 . However, if we use the Fourier expansion of f, we obtain $f(0)=1$. What is wrong?

5.3.3 Convergence of Fourier Series

In this section, we state the fundamental convergence theorem for Fourier Series, which assumes that the function $f(x)$ is piecewise continuous. At points of discontinuity of $f(x)$, the Fourier approximation $S_N(x)$ takes on the average value:

$$\frac{1}{2}\left(f\left(x^+\right) + f\left(x^-\right)\right)$$

where $f\left(x^+\right)$and $f\left(x^-\right)$ represent the right-hand and left-hand limits of $f(x)$ at the discontinuity, respectively. Before stating the main result of this section, we introduce the following notion.

Definition 5.3.3 (Piecewise continuous). *A function $f:[a,b]\to\mathbb{R}$ is piecewise continuous if there are numbers $t_0, t_1, \ldots, t_n$ with $a = t_0 < t_1 < \cdots < t_n = b$, such that f is continuous on each of the intervals (t_i, t_{i+1}), and tends to a finite value at each endpoint of these intervals. That is, the limits*

$$f\left(t_i^+\right) = \lim_{t \searrow t_i} f(t) \quad \text{and}$$
$$f\left(t_{i+1}^-\right) = \lim_{t \nearrow t_{i+1}} f(t)$$

exist (and are finite). A function $f : [a,b] \to \mathbb{R}$ *is piecewise continuous if, roughly speaking, it is made up of a finite number of continuous pieces. A function* $f : \mathbb{R} \to \mathbb{R}$ *is piecewise continuous if it is piecewise continuous on every closed interval* $[a,b]$*. Thus it can have infinitely many discontinuities, but only finitely many on any finite interval.*

Example 5.3.5.

- *For the square wave function*

$$f(t) = \begin{cases} 0 & \text{if} \quad -\pi < t \leq 0 \\ \pi & \text{if} \quad 0 < t \leq \pi \end{cases}$$

we have $f\left(0^-\right) = 0$ *and* $f\left(0^+\right) = \pi$ *and* f *is continuous on* $(-\pi, 0)$, $(0, \pi)$*. Therefore,* f *is piecewise continuous on* $[-\pi, \pi]$.

- *The function* $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(t) = \tan t$ *(and* $f(t) = 0$ *if* t *is an odd multiple of* $\pi/2$ *) is not piecewise continuous. Although it has only finitely many discontinuities on any finite interval, the function "blows up" at these discontinuities, so that (for example) the limits* $f\left(\frac{\pi}{2}^-\right)$ *and* $f\left(\frac{\pi}{2}^+\right)$ *do not exist.*

Theorem 5.3.1 (Convergence of Fourier Series). *Let* f *and* f' *be piecewise continuous functions on* $[-L, L]$ *(i.e.* f *is piecewise continuously differentiable or piecewise* C^1 *) and periodic with period* $2L$*, then* f *has a Fourier Series*

$$f(x)] \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = S(x)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \text{ and } b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

The Fourier Series converges to $f(x)$ *at all points at which* f *is continuous and to* $\frac{1}{2}\left[f\left(x^+\right) + f\left(x^-\right)\right]$ *at all points at which* f *is discontinuous. Thus a Fourier Series converges to the average value of the left and right limits at a point of discontinuity of the function* $f(x)$.

Important!

Under the hypothesis of Theorem 5.3.1, $\approx$ can be replaced by $=$ and

$$S(x) = \begin{cases} f(x), & \text{if } f \text{ is continuous at } x \\ \frac{f\left(x^+\right)+f\left(x^-\right)}{2}, & \text{if } f \text{ is discontinuous at the point } x. \end{cases}$$

Remark 5.3.1. *A way to look at the connection of Fourier series on an interval with the Fourier series of periodic functions is to start with a piecewise continuous function g defined only on the interval $[-L, L]$. Then g_{per} , a periodic extension of g of period $2L$, can play the role of f above; in particular, the Fourier series of g converges to g_{per} everywhere, in our usual sense:*

$$S(x) = \begin{cases} g_{\text{per}}(x), & \text{if } g_{\text{per}} \text{ is continuous at } x \\ \frac{g_{\text{per}}(x^+)+g_{\text{per}}(x^-)}{2}, & \text{if } g_{\text{per}} \text{ is discontinuous at the point } x \end{cases}$$

Remark 5.3.2 (Gibbs phenomenon). *The Fourier series has a difficult time converging at the point of discontinuity and these graphs of the Fourier series show a distinct overshoot which does not go away. This is called the Gibbs phenomenon and the amount of overshoot can be computed. We refer to the Lectures notes of Prof. Peirce for more on this phenomenon.*

Theorem 5.3.2 (Uniform convergence of Fourier Series). *Let f be a continuous functions on $[-L, L]$ and periodic with period $2L$. If f' is piecewise continuous on $[-L, L]$, then the Fourier series for f converges uniformly to f on $[-L, L]$ and hence on any interval. That is, for each $\varepsilon > 0$, there exists an integer N_0 (that depends on ε) such that*

$$\sup_{x\in[-L,L]} |f(x) - S_N(x)| < \varepsilon$$

for all $N > N_0$, where

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

5.3.4 Complex form of Fourier Series

Finally, everything said above applies also to the complex form of the Fourier series: a function $g(x)$, periodic with period $2L$, has a complex Fourier series

$$g(x) \approx \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{L}\right)}$$

with

$$c_n = \frac{1}{2L}\int_{-L}^{L} g(x)e^{-i\left(\frac{n\pi x}{L}\right)}dx$$

Example 5.3.6.

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 < x < \pi \end{cases} \qquad L = \pi$$

$$\begin{aligned} c_n &= \frac{1}{2\pi}\left\{ -\int_{-\pi}^{0} \mathrm{e}^{-inx} dx + \int_{0}^{\pi} \mathrm{e}^{-inx} dx \right\} \\ &= \frac{1}{2\pi}\left\{ -\frac{\mathrm{e}^{-inx}\big|_{-\pi}^{0}}{(-in)} + \frac{\mathrm{e}^{-inx}\big|_{0}^{\pi}}{(-in)} \right\} \\ &= \frac{i}{2\pi n}\left\{ -2 + \mathrm{e}^{+in\pi} + \mathrm{e}^{-in\pi} \right\} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{i\pi n} & n \text{ odd} \end{cases} \end{aligned}$$

Therefore,

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi i(2n+1)} \mathrm{e}^{i((2n+1)x)}$$

5.4 Bessel's inequality and Parseval Identity

Bessel's inequality and Parseval's identity are fundamental results in functional analysis and Fourier analysis, particularly in the study of Hilbert spaces. These results provide insights into the decomposition of functions into orthonormal bases and the convergence of series representations. We will explore an analogue of Pythagoras' Theorem for functions that are square-integrable. Such functions are significant in mathematical physics, as they correspond to systems with finite energy. Additionally, we show some applications of the Parseval's identity into summation formulas involving series of reciprocal powers of n.

Definition 5.4.1 (Square integrable function). *A function f is said to be square-integrable if it satisfies the condition:*

$$\int_{-L}^{L} [f(x)]^2 dx < \infty$$

in which case we write $f \in L^2([-L, L])$.

Consider the Fourier series associated with $f(x)$:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] = S_\infty$$

Define the partial sum:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Then, we have the following results.

Theorem 5.4.1 (Bessel's Inequality). *Let $f \in L^2[-L, L]$. Then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \frac{1}{L} \int_{-L}^{L} f^2(x) dx$$

in particular the series $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$ is convergent.

Proof of Theorem 5.4.1. We have that

$$[f(x) - S_N(x)]^2 = f^2(x) - 2f(x)S_N(x) + S_N^2(x)$$

Consider the least-square error defined as:

$$\begin{aligned} \mathcal{E}_2 [f, S_N] &= \frac{1}{L} \int_{-L}^{L} [f(x) - S_N(x)]^2 \, dx \\ &= \frac{1}{L} \left\{ \int_{-L}^{L} f^2(x) dx - 2 \int_{-L}^{L} f(x) S_N(x) dx + \int_{-L}^{L} S_N^2(x) dx \right\} \\ &= \frac{1}{L} \left\{ \langle f, f \rangle - 2 \langle f, S_N \rangle + \langle S_N, S_N \rangle \right\} \end{aligned}$$

Now, we compute:

$$\begin{aligned} \langle S_N, S_N \rangle &= \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]^2 dx \\ &= \frac{a_0^2}{2} L + \sum_{n=1}^{N} \left(a_n^2 \int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx + b_n^2 \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx \right) \\ &= L \left[\frac{a_0^2}{2} + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right) \right] \end{aligned}$$

In addition,

$$\begin{aligned} \langle f, S_N \rangle &= \int_{-L}^{L} f(x) S_N(x) dx \\ &= \frac{a_0}{2} \int_{-L}^{L} f(x) dx + \sum_{n=1}^{N} \left(a_n \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx + b_n \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right) \end{aligned}$$

Therefore,

$$\mathcal{E}_2 [f, S_N] = \frac{1}{L} \int_{-L}^{L} [f(x) - S_N(x)]^2 \, dx = \frac{1}{L} \langle f, f \rangle - \left\{ \frac{a_0^2}{2} + \sum_{n=1}^{N} \left(a_n^2 + b_n^2\right) \right\}$$

Since we know that

$$\mathcal{E}_2 [f, S_N] = \int_{-L}^{L} [f(x) - S_N(x)]^2 \, dx \geq 0$$

it follows that

$$\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2 \le \frac{1}{L}\int_{-L}^{L} f^2(x)dx = \frac{1}{L}\langle f, f\rangle = E[f]$$

where $E[f]$ is known as the energy of the $2L$-periodic function f.

□

Theorem 5.4.2 (Parseval's Identity). *Let $f \in L^2[-L, L]$. Then the Fourier coefficients a_n and b_n satisfy Parseval's Formula*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{L}\int_{-L}^{L} f^2(x)dx = E[f] \tag{5.78}$$

if and only if

$$\lim_{N\to\infty}\int_{-L}^{L}\left[f(x) - S_N(x)\right]^2 dx = 0 \tag{5.79}$$

Remark 5.4.1. *The convergence in (5.79) should be understood in the L^2-sense (mean square sense). This is a convergence in an average sense. When $\{S_N\}$ tends to f uniformly, $\{S_N\}$ must tend to f in L^2-sense. The converse is not always true. Hence convergence in L^2-sense is weaker than uniform convergence.*

Important!: In practice...

- The Fourier series of every L^2-integrable function converges to the function in L^2-sense.
- Let f be a piecewise continuous function on $[-L, L]$. Then S_N converges to f in the mean square sense.
- If f is piecewise continuous on $[-L, L]$, then Parseval's identity (5.78) holds.

Example 5.4.1. *Consider the Fourier cosine series of $f(x) = x$, $0 < x < 2$:*

$$x \approx 1 + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}[\cos(n\pi) - 1]\cos\frac{n\pi x}{2}$$

a) Write Parseval's identity corresponding to the above Fourier series.
b) Determine from a) the sum of the series

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

Solution 5.4.1 (Solution to Example 5.4.1)**. *a)*** *We first find the Fourier coefficient and the period of the Fourier series just by comparing the given series with the standard Fourier series*

$$a_0 = 2, \quad a_n = \frac{4}{\pi^2 n^2}[\cos(n\pi) - 1],\ n = 1, 2, \dots, \quad b_n = 0,$$

period: $L = 2$.

We note that

$$\int_{-2}^{2} x^2\,dx = \frac{16}{3} < \infty,$$

so that $f \in L^2(-2,2)$. Thus, the conditions for Parseval's identity are satisfied, and we can write

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{2}\int_{-2}^{2} f^2(x)\,dx = \int_0^2 f^2(x)\,dx.$$

This implies

$$\int_0^2 x^2\,dx = \frac{4}{2} + \sum_{n=1}^{\infty}\frac{16}{\pi^4 n^4}(\cos(n\pi) - 1)^2.$$

This can be simplified to give

$$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right],$$

and hence

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}.$$

b) *Let*

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots .$$

Split the series as

$$\begin{aligned} S &= \left(\tfrac{1}{1^4} + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \cdots\right) + \left(\tfrac{1}{2^4} + \tfrac{1}{4^4} + \tfrac{1}{6^4} + \cdots\right) \\ &= \frac{\pi^4}{96} + \frac{1}{2^4}S. \end{aligned}$$

Solving for S gives

$$S = \frac{\pi^4}{90}.$$

Example 5.4.2. *Find the Fourier series of $x^2, -\pi < x < \pi$ and use it along with Parseval's theorem to show that*

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}.$$

Solution 5.4.2 (Solution to Example 5.4.2)**.** *Since $f(x) = x^2$ is an even function, $b_n = 0$. For $n \geq 1$,*

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = \frac{2}{\pi}\left[0 - \frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx\right] = \frac{4}{n^2}(-1)^n.$$

The constant term is

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$

Thus

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx).$$

Since $\int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^5}{5} < \infty$*, by Parseval's theorem,*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

Using $\frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{2\pi^4}{5}$ *leads to*

$$\frac{4\pi^4}{18} + \sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{2\pi^4}{5},$$

so $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$*. Finally,*

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \sum_{n=1}^{\infty}\frac{1}{n^4} - \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{15}{16}\frac{\pi^4}{90} = \frac{\pi^4}{96}.$$

5.5 Heat conduction problems with time-independent inhomogeneous boundary conditions

In this section, we consider heat conduction problems with inhomogeneous boundary conditions. To determine a solution, we exploit the linearity of the problem, which ensures that linear combinations of solutions remain solutions. In particular, we first determine a well-chosen particular solution, known as the **steady-state solution**, which can be used to eliminate the inhomogeneous boundary conditions. This reduces the problem to solving the same boundary value problem but with homogeneous boundary conditions and an adjusted initial condition.

Selection of particular solution is not unique !!

Although the steady-state solution is a natural choice in this case, the selection of a particular solution, as always, is not unique. We will introduce two methods: The separation of variables and a more generally applicable method of eigenfunction expansions.

5.5.1 Steady state

We convert an inhomogeneous heat equation to a homogeneous problem when the inhomogeneous terms are all time-independent.

Dirichlet nonhomogeneous BC

Consider the Boundary Value Problem (BVP) modelling heat propagation in a rod where the end points are kept at constant temperatures u_0 and u_1 :

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, t > 0 \tag{5.80a}$$
$$\text{BC} : u(0,t) = u_0, \quad u(L,t) = u_1 \tag{5.80b}$$
$$\text{IC} : u(x,0) = f(x) \tag{5.80c}$$

Since u_0 and u_1 are not necessarily zero, we cannot apply directly the method of separation of variables. To solve such a problem, we can proceed as follows.

Methodology 5.5.1.

(a) Find the steady-state solution (i.e., when $u_t = 0$) which we denote by $u_\infty(x)$. (5.80) gives

$$u''_\infty(x) = 0 \Rightarrow u_\infty(x) = Ax + B$$

Using the boundary conditions, we obtain $u_\infty(0) = B = u_0$ and $u_\infty(L) = AL + B = u_1 \Rightarrow A = \frac{u_1 - u_0}{L}$. Therefore,

$$u_\infty(x) = \left(\frac{u_1 - u_0}{L}\right) x + u_0, \quad \text{(Steady-state solution)}. \tag{5.81}$$

(b) Let $v(x,t) = u(x,t) - u_\infty(x)$. We verify that if $u(x,t)$ solves the given BVP (5.80), then $v(x,t)$ solves the following homogeneous problem

$$v_t = \alpha^2 v_{xx}, \quad 0 < x < L, t > 0 \tag{5.82a}$$
$$\text{BC} : \quad v(0,t) = 0, \quad v(L,t) = 0 \tag{5.82b}$$
$$\text{IC} : \quad v(x,0) = f(x) - u_\infty(x) \tag{5.82c}$$

Indeed, we have:

$$v_t = u_t = \alpha^2 u_{xx} = \alpha^2 v_{xx}.$$

Hence, $v_t = \alpha^2 v_{xx}$. Moreover, $v(0,t) = u(0,t) - u_\infty(0) = u_0 - u_0 = 0$ and $v(L,t) = u(L,t) - u_\infty(L) = u_1 - u_1 = 0$. So, v solves (5.82).
Now, (5.82), has homogeneous BC and can be solved using the separation of method. Proceeding as in Section 5.2.1, we deduce that

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{5.83}$$

where

$$b_n = \frac{2}{L} \int_0^L (f(x) - u_\infty(x)) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots \tag{5.84}$$

Finally, the most general solution to (5.80) is given by

$$\boxed{u(x,t) = v(x,t) + u_\infty(x) = u_0 + \left(\frac{u_1 - u_0}{L}\right) x + \sum_{n=1}^{\infty} b_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right),} \tag{5.85}$$

with b_n given by(5.84).
Here, we also show a more general method known as the eigenfunction expansions.

Eigenfunction expansions

In order to solve the boundary value problem (5.82), we could recognize that $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ *are eigenfunctions of the spatial operator:*

$$-\frac{\partial^2}{\partial x^2}$$

along with the homogeneous Dirichlet BC $v(0,t) = 0 = v(L,t)$. *We therefore assume an eigenfunction expansion of the form:*

$$\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} \hat{v}_n(t)\sin\left(\frac{n\pi x}{L}\right)\\
v_t &= \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t)\sin\left(\frac{n\pi x}{L}\right) \text{ and}\\
v_{xx} &= -\sum_{n-1}^{\infty} \hat{v}_n(t)\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right)\\
v_t &= \alpha^2 v_{xx} \Rightarrow v_t - \alpha^2 v_{xx} = 0\\
&\Rightarrow \sum_{n=1}^{\infty}\left\{\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n(t)\right\}\sin\left(\frac{n\pi x}{L}\right) = 0
\end{aligned}$$

The sine functions $\sin\left(\frac{n\pi x}{L}\right)$ *form an orthogonal set over the interval* $x \in [0, L]$. *Therefore, the above equation holds if and only if each term in the summation is zero:*

$$\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n(t) = 0$$

This is a first-order linear ordinary differential equation for $\hat{v}_n(t)$. *The equation is separable and can be solved as follows:*

$$\dot{\hat{v}}_n(t) = -\alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n(t)$$

This has the general solution:

$$\hat{v}_n(t) = \hat{v}_n(0)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

where $\hat{v}_n(0)$ *is the initial condition for* $\hat{v}_n(t)$. *Therefore,*

$$v(x,t) = \sum_{n=1}^{\infty}\hat{v}_n(0)\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right).$$

We have

$$v(x,0) = \sum_{n=1}^{\infty}\hat{v}_n(0)\sin\left(\frac{n\pi x}{L}\right) = f(x) - u_\infty(x)$$

Hence by projection,

$$\hat{v}_n(0) = \frac{2}{L}\int_0^L \{f(x) - u_\infty(x)\}\sin\left(\frac{n\pi x}{L}\right)dx$$

which is the same solution as above.

Example 5.5.1. *We aim to solve the heat conduction problem:*

$$\begin{cases} u_t = u_{xx}, & 0 < x < 2, \quad t > 0 \\ u(0,t) = 100, \quad u(2,t) = 100, & t > 0 \\ u(x,0) = 0, & 0 < x < 2 \end{cases} \tag{5.86}$$

The steady state function $u_\infty(x)$ satisfies the equation:

$$u''_\infty(x) = 0, \quad u_\infty(0) = 100, \quad u_\infty(2) = 100$$

Solving this, we obtain:

$$u_\infty(x) = 100$$

Define $v(x,t) = u(x,t) - u_\infty(x)$. Then v satisfies the following boundary value problem:

$$\begin{cases} v_t = v_{xx}, & 0 < x < 2, \quad t > 0 \\ v(0,t) = 0, \quad v(2,t) = 0, & t > 0 \\ v(x,0) = -100, & 0 < x < 2 \end{cases}$$

The solutions of the homogeneous part, using separation of variables, are given by:

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{\pi^2 n^2}{4} t} \sin \frac{n\pi x}{2}$$

The initial condition implies:

$$v(x,0) = -100 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{2}$$

The Fourier coefficients are computed as:

$$C_n = \int_0^2 (-100) \sin \frac{n\pi x}{2} dx = \frac{200\,[(-1)^n - 1]}{n\pi}$$

Thus, we obtain:

$$v(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n} e^{-\frac{\pi^2 n^2}{4} t} \sin \frac{n\pi x}{2}$$

Rewriting in terms of odd indices:

$$v(x,t) = -\frac{400}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} e^{-\frac{\pi^2 (2k+1)^2}{4} t} \sin \frac{(2k+1)\pi x}{2}$$

Finally, the solution to the original problem is:

$$\boxed{u(x,t) = 100 - \frac{400}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} e^{-\frac{\pi^2 (2k+1)^2}{4} t} \sin \frac{(2k+1)\pi x}{2}}$$

Mixed Dirichlet-Neumann nonhomogeneous BC

We consider the initial-boundary value problem

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{5.87a}$$

$$\text{BC}: \quad u(0,t) = u_0, \quad u_x(L,t) = q_1 \tag{5.87b}$$

$$\text{IC}: \quad u(x,0) = g(x) \tag{5.87c}$$

Methodology 5.5.2.

(a) Steady-State Solution. *For the steady state we set* $u_t = 0$*, so that*

$$u''_\infty(x) = 0$$

Integrating twice, we obtain

$$u_\infty(x) = Ax + B.$$

The steady state must satisfy the boundary conditions. Since

$$u_\infty(0) = B = u_0$$

and, because the boundary condition at $x = L$ *is on the derivative,*

$$u'_\infty(L) = A = q_1$$

we deduce that

$$u_\infty(x) = u_0 + q_1 x. \quad (\textit{ Steady-state solution }) \tag{5.88}$$

(b) Reduction to a Homogeneous Problem. *Define*

$$v(x,t) = u(x,t) - u_\infty(x)$$

Since $u_\infty(x)$ *is independent of* t*, we have*

$$v_t = u_t \quad \textit{and} \quad v_{xx} = u_{xx}$$

Thus, $v(x,t)$ *satisfies*

$$v_t = \alpha^2 v_{xx}, \quad 0 < x < L, \ t > 0$$

Moreover, the boundary conditions transform as follows:

$$v(0,t) = u(0,t) - u_\infty(0) = u_0 - u_0 = 0$$
$$v_x(L,t) = u_x(L,t) - u'_\infty(L) = q_1 - q_1 = 0$$

And the initial condition becomes

$$v(x,0) = g(x) - u_\infty(x) = g(x) - (u_0 + u_1 x)$$

Thus, $v(x,t)$ satisfies the homogeneous problem

$$v_t = \alpha^2 v_{xx}, \quad 0 < x < L, t > 0 \tag{5.89a}$$
$$v(0,t) = 0, \quad v_x(L,t) = 0 \tag{5.89b}$$
$$v(x,0) = g(x) - (u_0 + u_1 x) \tag{5.89c}$$

(c) Separation of Variables. *Assume a solution of the form*

$$v(x,t) = X(x)T(t).$$

Substitute into the PDE in (5.89):

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Dividing by $\alpha^2 X(x)T(t)$, we obtain

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \lambda,$$

where λ is the separation constant.
In order to have a decaying solution in time we set

$$\lambda = -\mu^2, \quad \mu > 0.$$

Then the spatial ODE becomes

$$X''(x) + \mu^2 X(x) = 0 \tag{5.90}$$

with the boundary conditions

$$X(0) = 0, \quad X'(L) = 0$$

The general solution of (5.90) is

$$X(x) = A\cos(\mu x) + B\sin(\mu x).$$

The condition $X(0) = 0$ forces $A = 0$, hence

$$X(x) = B\sin(\mu x)$$

Enforcing the Neumann condition at $x = L$,

$$X'(x) = B\mu\cos(\mu x), \quad X'(L) = B\mu\cos(\mu L) = 0$$

we require (for nontrivial B)

$$\cos(\mu L) = 0$$

This implies

$$\mu L = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \ldots$$

so that

$$\mu_n = \frac{(2n+1)\pi}{2L} \quad \text{and} \quad \lambda_n = -\mu_n^2 = -\left(\frac{(2n+1)\pi}{2L}\right)^2.$$

Thus, the eigenfunctions are

$$X_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

The time-dependent ODE becomes

$$T'(t) = \alpha^2 \lambda_n T(t) = -\alpha^2 \mu_n^2 T(t),$$

which has the solution

$$T_n(t) = e^{-\alpha^2 \mu_n^2 t}$$

(d) Series Solution. *By superposition, the solution to (5.89) is given by*

$$v(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) e^{-\alpha^2 \left(\frac{(2n+1)\pi}{2L}\right)^2 t} \tag{5.91}$$

where the Fourier coefficients are determined from the initial condition:

$$b_n = \frac{2}{L} \int_0^L [g(x) - (u_0 + u_1 x)] \sin\left(\frac{(2n+1)\pi x}{2L}\right) dx. \tag{5.92}$$

(e) Final Solution. *Returning to the original variable, we have*

$$u(x,t) = v(x,t) + u_\infty(x).$$

That is,

$$\boxed{u(x,t) = u_0 + u_1 x + \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) e^{-\alpha^2 \left(\frac{(2n+1)\pi}{2L}\right)^2 t}}$$

with the coefficients b_n given in (5.92).

5.5.2 Heat conduction with some heat loss and inhomogeneous boundary conditions

We consider the initial-boundary value problem

$$u_t = \alpha^2 u_{xx} - u, \quad 0 < x < L, t > 0 \tag{5.93a}$$

$$\text{BC}: \quad u(0,t) = 0, \quad u(L,t) = u_1 \tag{5.93b}$$

$$\text{IC}: \quad u(x,0) = g(x) \tag{5.93c}$$

Methodology 5.5.3.

(a) Steady-State Solution. *To find the steady-state solution $u_\infty(x)$, we set $u_t = 0$ in (5.93) so that*

$$\alpha^2 u_\infty''(x) - u_\infty(x) = 0.$$

This ODE can be rewritten as

$$u_\infty''(x) - \frac{1}{\alpha^2} u_\infty(x) = 0.$$

Its characteristic equation is

$$r^2 - \frac{1}{\alpha^2} = 0,$$

with roots

$$r = \pm \frac{1}{\alpha}.$$

Hence, the general solution is

$$u_\infty(x) = C_1 e^{x/\alpha} + C_2 e^{-x/\alpha}$$

Applying the boundary condition at $x = 0$:

$$u_\infty(0) = C_1 + C_2 = 0 \quad \Longrightarrow \quad C_2 = -C_1$$

so that

$$u_\infty(x) = C_1 \left(e^{x/\alpha} - e^{-x/\alpha}\right) = 2C_1 \sinh\left(\frac{x}{\alpha}\right)$$

Next, using the condition at $x = L$:

$$u_\infty(L) = 2C_1 \sinh\left(\frac{L}{\alpha}\right) = u_1$$

we find

$$C_1 = \frac{u_1}{2\sinh(L/\alpha)}$$

Thus, the steady-state solution is

$$u_\infty(x) = \frac{u_1 \sinh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)} \tag{5.94}$$

(b) Reduction to a Homogeneous Problem. *Define the transient variable*

$$v(x,t) = u(x,t) - u_\infty(x)$$

Since $u_\infty(x)$ is independent of t, we have

$$v_t = u_t \quad \text{and} \quad v_{xx} = u_{xx} - u''_\infty(x)$$

Substituting $u(x,t) = v(x,t) + u_\infty(x)$ into the PDE (5.93) yields

$$v_t = \alpha^2 \left(v_{xx} + u''_\infty(x)\right) - \left(v + u_\infty(x)\right).$$

But $u_\infty(x)$ satisfies

$$\alpha^2 u''_\infty(x) - u_\infty(x) = 0$$

so that the transient variable satisfies

$$v_t = \alpha^2 v_{xx} - v$$

The boundary conditions become

$$\begin{aligned} v(0,t) &= u(0,t) - u_\infty(0) = 0 - 0 = 0 \\ v(L,t) &= u(L,t) - u_\infty(L) = u_1 - u_1 = 0 \end{aligned}$$

and the initial condition is

$$v(x,0) = g(x) - u_\infty(x) = g(x) - \frac{u_1 \sinh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)}$$

Therefore, $v(x,t)$ satisfies

$$v_t = \alpha^2 v_{xx} - v, \quad 0 < x < L, t > 0, \tag{5.95a}$$

$$v(0,t) = 0, \quad v(L,t) = 0, \tag{5.95b}$$

$$v(x,0) = g(x) - \frac{u_1 \sinh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)}. \tag{5.95c}$$

(c) Separation of Variables. *Rewrite the PDE for $v(x,t)$ as*

$$v_t + v = \alpha^2 v_{xx}$$

Assume a solution of the form

$$v(x,t) = X(x)T(t).$$

Substituting into the equation gives

$$X(x)T'(t) + X(x)T(t) = \alpha^2 X''(x)T(t)$$

Dividing by $\alpha^2 X(x)T(t)$ (assuming nonzero factors) leads to

$$\frac{T'(t)}{\alpha^2 T(t)} + \frac{1}{\alpha^2} = \frac{X''(x)}{X(x)}$$

Since the left side depends only on t and the right only on x, we set them equal to a constant $-\mu^2$:

$$\frac{T'(t)}{\alpha^2 T(t)} + \frac{1}{\alpha^2} = -\mu^2$$

Multiplying by α^2 yields

$$\frac{T'(t)}{T(t)} + 1 = -\alpha^2\mu^2$$

Hence, the timedependent equation is

$$T'(t) = \left(-\alpha^2\mu^2 - 1\right)T(t) \quad \Longrightarrow \quad T(t) = e^{\left(-\alpha^2\mu^2-1\right)t}$$

The spatial part satisfies

$$\frac{X''(x)}{X(x)} = -\mu^2 \quad \Longrightarrow \quad X''(x) + \mu^2 X(x) = 0$$

With the boundary conditions $X(0) = 0$ and $X(L) = 0$ the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \mu = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

Thus, for each n we have

$$T_n(t) = e^{-t-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

(d) Series Representation and Final Solution. *By superposition, the solution to the homogeneous problem for $v(x,t)$ is given by*

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-t-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

where the Fourier sine coefficients are determined from the initial condition:

$$b_n = \frac{2}{L}\int_0^L \left[g(x) - \frac{u_1 \sinh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)}\right] \sin\left(\frac{n\pi x}{L}\right) dx$$

Returning to the original variable, the full solution is

$$\boxed{u(x,t) = u_\infty(x) + v(x,t) = \frac{u_1 \sinh\left(\frac{x}{\alpha}\right)}{\sinh\left(\frac{L}{\alpha}\right)} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-t-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}}$$

where b_n are given above.

5.5.3 Heat Conduction problems with distributed time-independent sources

We consider the initial-boundary value problem

$$u_t = \alpha^2 u_{xx} + x, \quad 0 < x < L, t > 0, \tag{5.96a}$$

$$\text{BC}: \quad u(0,t) = 0, \quad u(L,t) = B, \tag{5.96b}$$

$$\text{IC}: \quad u(x,0) = g(x). \tag{5.96c}$$

Methodology 5.5.4.

(a) Steady-State Solution. *To find the steady-state solution $u_\infty(x)$, we set $u_t = 0$ in (5.96), leading to the ODE*

$$\alpha^2 u''_\infty(x) + x = 0.$$

Integrating twice, we obtain

$$\alpha^2 u'_\infty(x) = -\frac{x^2}{2} + C_1,$$

$$u_\infty(x) = -\frac{x^3}{6\alpha^2} + C_1 x + C_2.$$

Applying the boundary conditions $u_\infty(0) = 0$ and $u_\infty(L) = B$, we get

$$C_2 = 0$$

$$-\frac{L^3}{6\alpha^2} + C_1 L = B$$

Solving for C_1, we find

$$C_1 = \frac{B + L^3/6\alpha^2}{L}$$

Thus, the steady-state solution is

$$u_\infty(x) = -\frac{x^3}{6\alpha^2} + \left(\frac{B + L^3/6\alpha^2}{L}\right)x$$

(b) Reduction to a Homogeneous Problem. *Define the transient variable*

$$v(x,t) = u(x,t) - u_\infty(x)$$

Since $u_\infty(x)$ is independent of t, we have

$$v_t = u_t, \quad v_{xx} = u_{xx} - u_\infty''(x)$$

Substituting $u(x,t) = v(x,t) + u_\infty(x)$ into (5.96) gives

$$v_t = \alpha^2\left(v_{xx} + u_\infty''(x)\right) + x$$

But we know that $u_\infty''(x) = -x/\alpha^2$, so

$$v_t = \alpha^2 v_{xx} + x - x = \alpha^2 v_{xx}$$

The boundary conditions remain homogeneous:

$$v(0,t) = u(0,t) - u_\infty(0) = 0, \quad v(L,t) = u(L,t) - u_\infty(L) = 0$$

The initial condition is

$$v(x,0) = g(x) - u_\infty(x)$$

(c) Separation of Variables. *Assume a solution of the form*

$$v(x,t) = X(x)T(t)$$

Substituting into the homogeneous equation $v_t = \alpha^2 v_{xx}$ gives

$$X(x)T'(t) = \alpha^2 X''(x)T(t)$$

Dividing by $X(x)T(t)$ leads to

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$$

Since the left-hand side depends only on t and the right-hand side only on x, both must equal a constant λ :

$$\frac{T'(t)}{\alpha^2 T(t)} = \lambda, \quad T'(t) = \alpha^2 \lambda T(t)$$

Solving for $T(t)$, we get

$$T(t) = e^{\alpha^2 \lambda t}$$

The spatial equation becomes

$$X''(x) = \lambda X(x)$$

For nontrivial solutions under homogeneous Dirichlet conditions $X(0) = X(L) = 0$, we set

$$\lambda = -k^2, \quad k = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

The eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

and the separation constants are

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

Hence, the time-dependent part is

$$T_n(t) = e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$

(d) Series Representation and Final Solution. *By superposition, the solution to the homogeneous problem for $v(x,t)$ is given by*

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$

where the Fourier sine coefficients are determined from the initial condition:

$$b_n = \frac{2}{L} \int_0^L [g(x) - u_\infty(x)] \sin\left(\frac{n\pi x}{L}\right) dx$$

Finally, recalling that

$$u(x,t) = v(x,t) + u_\infty(x)$$

we obtain the full solution:

$$\boxed{u(x,t) = -\frac{x^3}{6\alpha^2} + \left(\frac{B + L^3/6\alpha^2}{L}\right) x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}}$$

where b_n are given as above. We remark that

$$\lim_{t\to\infty} u(x,t) = x\left\{\frac{B}{L} + \frac{1}{6\alpha^2}\left(L^2 - x^2\right)\right\}$$

More generally, if we consider the problem:

$$u_t = \alpha^2 u_{xx} + h(x), \quad 0 < x < L, t > 0, \tag{5.97a}$$

$$\text{BC}: \quad u(0,t) = A, \quad u(L,t) = B, \tag{5.97b}$$

$$\text{IC}: \quad u(x,0) = f(x), \tag{5.97c}$$

which represents heat flow with a time-independent source and/or ends fixed at some temperature. The expectation is that over time, the heat will approach a steady state (equilibrium): $\bar{u}(x) = \lim_{t\to\infty} u(x,t)$.

Formally, we can obtain this equilibrium shape as follows: if $\bar{u}(x)$ is a steady-state, then it solves the PDE but does not depend on time. Thus it must satisfy

$$0 = \alpha^2 \bar{u}_{xx} + h(x), \quad \bar{u}(0) = A, \quad \bar{u}(L) = B$$

The important point is that the difference between the PDE solution and steady state,

$$v = u - \bar{u}$$

solves the homogeneous problem

$$v_t = \alpha^2 v_{xx}, \quad 0 < x < L, t > 0 \tag{5.98a}$$

$$\text{BC}: \quad v(0,t) = 0, \quad v(L,t) = 0 \tag{5.98b}$$

$$\text{IC}: \quad u(x,0) = f(x) - \bar{u}(x) \tag{5.98c}$$

Note

So to solve (5.97) we can find the steady state (formally), subtract it out and then solve (5.98) for the "homogeneous" part.

Particular solution

In some cases, the previous method may fail. This trick works when there is a steady state and only when the source term and boundary conditions do not depend on time. For instance,

$$u_t = t u_{xx} + \sin x$$

cannot be solved using this method. Assuming $u_t = 0$ is not enough since we also need to take $t \to \infty$ and we cannot find a $u = \bar{u}(x)$ that solves

$$t u_{xx} + \sin x = 0.$$

We can, however, guess a particular solution of the equation or more generally solve the full problem using the eigenfunction method.

5.5.4 Neumann nonhomogeneous BC

We wish to solve

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < L, t > 0 \tag{5.99a}$$
$$\text{BC}: \quad u_x(0,t) = A, \quad u_x(L,t) = B \tag{5.99b}$$
$$\text{IC}: \quad u(x,0) = g(x) \tag{5.99c}$$

Methodology 5.5.5.

(a) Finding a Particular Solution. *Try for a steady solution: $u''_\infty(x) = 0$, $u_\infty(x) = ax + b$, $u_x = a$ but then we cannot match both BC unless $A = B = a$. This means that if we are pumping and removing heat from the rod at different rates then the temperature does not reach a steady state.*

Since the boundary conditions are given on the spatial derivative, we first look for a function $\phi(x,t)$ that satisfies both the PDE and the nonhomogeneous Neumann conditions.

We assume a solution of the form

$$\phi(x,t) = ax^2 + bx + c + dt,$$

where a, b, c, d are constants (possibly depending on the data). We have

$$\phi_t = d \quad \text{and} \quad \phi_{xx} = 2a$$

Substituting into the PDE $\phi_t = \alpha^2\phi_{xx}$ *gives*

$$d = \alpha^2(2a) \quad \Longrightarrow \quad a = \frac{d}{2\alpha^2}$$

Next, we require that ϕ satisfies the boundary conditions on the derivative. Since

$$\phi_x(x,t) = 2ax + b$$

we impose

$$\phi_x(0,t) = b = A$$

and at $x = L$

$$\phi_x(L,t) = 2aL + b = B$$

Thus,

$$2aL = B - A \quad \Longrightarrow \quad a = \frac{B-A}{2L}.$$

Then, from $a = \frac{d}{2\alpha^2}$ we deduce

$$d = \alpha^2\frac{B-A}{L}.$$

We may choose $c = 0$ for simplicity. Hence, a particular solution is given by

$$\phi(x,t) = \frac{B-A}{2L}x^2 + Ax + \frac{\alpha^2(B-A)}{L}t \tag{5.100}$$

(b) Reduction to a Homogeneous Problem. *Define*

$$v(x,t) = u(x,t) - \phi(x,t)$$

Since both u and ϕ satisfy the heat equation, it follows that

$$v_t = u_t - \phi_t = \alpha^2 u_{xx} - \alpha^2 \phi_{xx} = \alpha^2 v_{xx}$$

The boundary conditions for v become

$$\begin{aligned} v_x(0,t) &= u_x(0,t) - \phi_x(0,t) = A - A = 0, \\ v_x(L,t) &= u_x(L,t) - \phi_x(L,t) = B - B = 0. \end{aligned}$$

The initial condition is

$$v(x,0) = u(x,0) - \phi(x,0) = g(x) - \left[\frac{B-A}{2L}x^2 + Ax\right]$$

Thus, $v(x,t)$ satisfies the homogeneous problem

$$v_t = \alpha^2 v_{xx}, \quad 0 < x < L, t > 0 \tag{5.101a}$$

$$v_x(0,t) = 0, \quad v_x(L,t) = 0 \tag{5.101b}$$

$$v(x,0) = g(x) - \left[\frac{B-A}{2L}x^2 + Ax\right] \tag{5.101c}$$

(c) Recall that the solution to (5.101) can be written as

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t},$$

where the Fourier coefficients are determined from

$$a_n = \frac{2}{L}\int_0^L \left[g(x) - \left(\frac{B-A}{2L}x^2 + Ax\right)\right] \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, \ldots$$

(d) Final Solution. *Recalling that*

$$u(x,t) = v(x,t) + \phi(x,t)$$

with $\phi(x,t)$ given in (5.100), we obtain the full solution

$$\boxed{u(x,t) = \frac{B-A}{2L}x^2 + Ax + \frac{\alpha^2(B-A)}{L}t + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t},}$$

where the coefficients a_n are given above.

5.5.5 Solving the non-homgeneous heat equation with homogeneous BC

The aim of this section is to solve the following non-homogeneous heat equation:

$$\begin{aligned} u_t = \alpha^2 u_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u(0,t) = 0 \text{ and } u(L,t) = 0 \text{ for } t > 0 \\ u(x,0) = f(x) \text{ for } 0 \leq x \leq L \end{aligned} \tag{5.102}$$

where $F(x,t)$ represents a source of heat energy in the medium.

Methodology 5.5.6. *We know that the eigenvalues and eigenfunctions associated with Dirichlet homogeneous BC are*

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \dots \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

For the non-homogeneous problem, we will take a cue from that case and attempt a solution

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{5.103}$$

The problem is to determine each $T_n(t)$. We observe that, for a given t, equation (5.103) can be interpreted as the Fourier sine expansion of $u(x,t)$, considered as a function of x, with $T_n(t)$ being the nth Fourier coefficient in this expansion, carefully establish the expression for $T_n(t)$. Therefore, by projecting $u(x,t)$ on the basis $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}$ and using orthogonality relations, we can deduce that

$$T_n(t) = \frac{2}{L}\int_0^L u(\xi,t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi \tag{5.104}$$

Next, we assume that for each $t \geq 0$, $F(x,t)$, as a function of x, can also be expanded in a Fourier sine series:

$$F(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right) dx$$

where

$$B_n(t) = \frac{2}{L}\int_0^L F(\xi,t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi \tag{5.105}$$

is the coefficient in this expansion, and may depend on t. We differentiate equation (5.104) to obtain

$$T_n'(t) = \frac{2}{L}\int_0^L u_t(\xi,t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

Since $u_t = \alpha^2 u_{xx} + F(x,t)$, the previous equation becomes

$$T_n'(t) = \frac{2\alpha^2}{L}\int_0^L u_{xx}(\xi,t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi + \frac{2}{L}\int_0^L F(\xi,t) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

In view of equation (5.105),

$$T_n'(t) = \frac{2\alpha^2}{L}\int_0^L u_{xx}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi + B_n(t) \tag{5.106}$$

Apply integration by parts twice to the integral in equation (5.106), using the boundary conditions and, at the last step, we obtain:

$$\begin{aligned}
\int_0^L u_{xx}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi &= \left[\sin\left(\frac{n\pi\xi}{L}\right)u_x(\xi,t)\right]_0^L - \int_0^L u_x(\xi,t)\frac{n\pi}{L}\cos\left(\frac{n\pi\xi}{L}\right)d\xi \\
&= -\int_0^L \frac{n\pi}{L}u_x(\xi,t)\cos\left(\frac{n\pi\xi}{L}\right)d\xi \\
&= \left[-\frac{n\pi}{L}u(\xi,t)\cos\left(\frac{n\pi\xi}{L}\right)\right]_0^L + \frac{n\pi}{L}\int_0^L u(\xi,t)\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)d\xi \\
&= -\frac{n^2\pi^2}{L^2}\int_0^L u(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi \\
&= -\frac{n^2\pi^2}{L^2}\cdot\frac{L}{2}T_n(t) \\
&= -\frac{n^2\pi^2}{2L}T_n(t)
\end{aligned}$$

Substituting this into equation (5.106) yields

$$T_n'(t) = -\alpha^2\frac{n^2\pi^2}{L^2}T_n(t) + B_n(t)$$

For $n = 1, 2, \cdots$, we now have an ordinary differential equation for $T_n(t)$:

$$T_n' + \alpha^2\frac{n^2\pi^2}{L^2}T_n = B_n(t)$$

Next, use equation (5.104) to obtain the condition,

$$\begin{aligned}
T_n(0) &= \frac{2}{L}\int_0^L u(\xi,0)\sin\left(\frac{n\pi\xi}{L}\right)d\xi \\
&= \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi = b_n
\end{aligned}$$

the n^{th} coefficient in the Fourier sine expansion of f on $[0, L]$. We solve the ordinary differential equation subject to the condition $T_n(0) = b_n$ to obtain the unique solution

$$T_n(t) = \int_0^t e^{-\alpha^2n^2\pi^2(t-\tau)/L^2}B_n(\tau)d\tau + b_ne^{-\alpha^2n^2\pi^2t/L^2}$$

Finally, substitute this into equation (5.103) to obtain

$$\begin{aligned}
u(x,t) = \sum_{n=1}^{\infty}&\left(\int_0^t e^{-\alpha^2n^2\pi^2(t-\tau)/L^2}B_n(\tau)d\tau\right)\sin\left(\frac{n\pi x}{L}\right) \\
&+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha^2n^2\pi^2t/L^2}
\end{aligned} \tag{5.107}$$

where $b_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$ and B_n is given.

Example 5.5.2. *Solve the following partial differential equation:*

$$\begin{cases} u_t = 4u_{xx} + t^2\cos(x/2), & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & t \geq 0 \\ u(x,0) = \begin{cases} 0, & 0 \leq x \leq \pi/2, \\ 25, & \pi/2 < x < \pi. \end{cases} \end{cases} \tag{5.108}$$

We let $F(x,t) = t^2\cos(x/2)$ and $L = \pi$. Following the methodology given above, we compute

$$B_n(t) = \frac{2}{\pi}\int_0^\pi t^2\cos(\xi/2)\sin(n\xi)d\xi = \frac{8}{\pi}\frac{2n}{4n^2-1}t^2$$

Now we can evaluate the integral occurring in the first summation of equation (5.107):

$$\int_0^t \frac{8}{\pi}\frac{2n}{4n^2-1}\tau^2 e^{-4n^2(t-\tau)}d\tau = \frac{1}{2}\frac{-4n^2t + 8n^4t^2 + 1 - e^{-4n^2t}}{n^5\pi\left(4n^2-1\right)}$$

Finally, compute

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(\xi)\sin(n\xi)d\xi \\ &= \frac{2}{\pi}\int_{\pi/2}^\pi 25\sin(n\xi)d\xi = \frac{50}{n\pi}\left(\cos(n\pi/2) - (-1)^n\right) \end{aligned}$$

The solution is

$$\begin{aligned} u(x,t) = &\sum_{n=1}^\infty \left(\frac{1}{2}\frac{-4n^2t + 8n^4t^2 + 1 - e^{-4n^2t}}{n^5\pi\left(4n^2-1\right)}\right)\sin(nx) \\ &+ \sum_{n=1}^\infty \frac{50}{n\pi}\left(\cos(n\pi/2) - (-1)^n\right)\sin(nx)e^{-4n^2t} \end{aligned}$$

Remark 5.5.1. *We note that the second summation in $u(x,t)$ is the solution of the initial-boundary value problem if the source term $t^2\cos(x/2)$ is omitted. If we denote this solution as $u_h(x,t)$ (the subscript h is for the fact that the heat equation without $F(x,t)$ is homogeneous), then*

$$u_h(x,t) = \sum_{n=1}^\infty \frac{50}{n\pi}\left(\cos(n\pi/2) - (-1)^n\right)\sin(nx)e^{-4n^2t}$$

and

$$u(x,t) = u_h(x,t) + \sum_{n=1}^\infty \left(\frac{1}{2}\frac{-4n^2t + 8n^4t^2 + 1 - e^{-4n^2t}}{n^5\pi\left(4n^2-1\right)}\right)\sin(nx)$$

This way of writing the solution clarifies which terms in the solution arise from the $t^2\cos(x/2)$ term in the partial differential equation.

5.6 Solving the non-homgeneous heat equation with nonhomogeneous time dependent BC

Now, we aim to solve the following non-homogeneous heat equation:

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= \phi_0(t) \text{ and } u(L,t) = \phi_1(t) \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 \le x \le L. \end{aligned} \tag{5.109}$$

Methodology 5.6.1. *It is too much to try to find a simple function that satisfies the PDE and the BCs simultaneously. However, we can instead compromise and find a function w that satisfies the boundary conditions but not the PDE. There are many choices for such a function w. For Dirichlet BCs, the easiest is just to construct a line that goes from $(0, \phi_0(t))$ to $(L, \phi_1(t))$:*

$$w(x,t) = \phi_0(t) + x\left(\frac{\phi_1(t) - \phi_0(t)}{L}\right)$$

Now let $u(x,t) = w(x,t) + v(x,t)$. Then, $v(x,t)$ is the solution of the following:

$$\begin{aligned} v_t = c^2 v_{xx} + F(x,t) - \dot{\phi}_0(t) - x\left(\tfrac{\dot{\phi}_1(t) - \dot{\phi}_0(t)}{L}\right), &\quad t > 0, \quad 0 < x < L \\ v(0,t) = 0, \quad v(L,t) = 0, &\quad t > 0 \\ v(x,0) = f(x) - \phi_0(0) - x\left(\tfrac{\phi_1(0) - \phi_0(0)}{L}\right) & \end{aligned} \tag{5.110}$$

We can use the eigenfunction expansions method as before to solve the transformed system.

Example 5.6.1. *Solve the following PDE with non-homogeneous BC*

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \textit{ for } 0 < x < L, t > 0 \\ u(0,t) &= At \textit{ and } u(L,t) = 0 \textit{ for } t > 0 \\ u(x,0) &= 0 \textit{ for } 0 \le x \le L \end{aligned} \tag{5.111}$$

In this case, $w(x,t) = At + \frac{x}{L}(0 - At) = At\left(1 - \frac{x}{L}\right)$. Next let $u(x,t) = w(x,t) + v(x,t)$, then

$$\begin{aligned} v_t &= \alpha^2 v_{xx} - A\left(1 - \tfrac{x}{L}\right) \\ v(0,t) &= 0 = v(L,t) \\ v(x,0) &= 0 \end{aligned}$$

We know that the eigenvalues and eigenfunctions associated with Dirichlet homogeneous BC are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, \ldots \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

So we let

$$v(x,t) = \sum_{n=1}^{\infty} \hat{v}_n(t) \sin\left(\frac{n\pi x}{L}\right)$$
$$v_t = \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t) \sin\left(\frac{n\pi x}{L}\right), \quad v_{xx} = -\sum_{n=1}^{\infty} \hat{v}_n(t) \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right)$$

Moreover, we write

$$s(x,t) = -A\left(1-\frac{x}{L}\right) = \sum_{n=1}^{\infty} \hat{s}_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

where

$$\hat{s}_n = \frac{2}{L}\int_0^L A\left(\frac{x}{L}-1\right)\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2A}{n\pi}$$

In addition, the initial condition gives $\hat{v}_n(0) = 0$. Therefore,

$$0 = v_t - \alpha^2 v_{xx} - s(x,t) = \sum_{n=1}^{\infty}\left\{\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n + \frac{2A}{n\pi}\right\}\sin\left(\frac{n\pi x}{L}\right)$$

Since $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}$ form a basis, we deduce that

$$\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n(t) = -\frac{2A}{n\pi}$$

This is a first-order linear ordinary differential equation. We can solve it using the integrating factor method. The equation becomes:

$$\frac{d}{dt}\left(e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n(t)\right) = -\frac{2A}{n\pi}e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

Now, integrating both sides:

$$e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2(n\pi)^3}e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t} + B_n$$

Here, B_n is the constant of integration. Solving for $\hat{v}_n(t)$, we get:

$$\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2(n\pi)^3} + B_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

To determine B_n, we use the initial condition $\hat{v}_n(0) = 0$:

$$0 = \hat{v}_n(0) = -\frac{2AL^2}{\alpha^2(n\pi)^3} + B_n$$

Solving for B_n, we find:

$$B_n = \frac{2AL^2}{\alpha^2(n\pi)^3}$$

Thus, the solution for $\hat{v}_n(t)$ becomes:

$$\hat{v}_n(t) = \frac{2AL^2}{\alpha^2(n\pi)^3}\left(e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} - 1\right)$$

Finally, the solution for $u(x,t)$ is obtained by summing over all modes n :

$$\boxed{u(x,t) = At\left(1-\frac{x}{L}\right) + \frac{2AL^2}{\pi^3\alpha^2}\sum_{n=1}^{\infty}\frac{\left(e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} - 1\right)}{n^3}\sin\left(\frac{n\pi x}{L}\right)}$$

Category	Dirichlet	Neumann	Periodic
Boundary Conditions	$\phi(0)=0,\ \phi(L)=0$	$\phi'(0)=0,\ \phi'(L)=0$	$\phi(-L)=\phi(L),\ \phi'(-L)=\phi'(L)$
Eigenvalues	$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\ n=1,2,3,\ldots$	$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\ n=0,1,2,\ldots$	$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\ n=0,1,2,\ldots$
Eigenfunctions	$\phi_n(x)=\sin\left(\frac{n\pi x}{L}\right)$	$\phi_n(x)=\cos\left(\frac{n\pi x}{L}\right)$	Both $\phi_n(x)=\begin{cases}\cos\left(\frac{n\pi x}{L}\right),\\ \sin\left(\frac{n\pi x}{L}\right)\end{cases}$
Series Expansion	$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)$	$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right]$
Coefficients	$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$ $b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Table 5.1: Boundary value problem for the equation $\frac{d^2\phi}{dx^2}=-\lambda\phi$

Category	Mixed (Dirichlet at $x=0$, Neumann at $x=L$)	Mixed (Neumann at $x=0$, Dirichlet at $x=L$)
Boundary Conditions	$\phi(0)=0,\ \phi'(L)=0$	$\phi'(0)=0,\ \phi(L)=0$
Eigenvalues	$\lambda_n=\left(\frac{(2n+1)\pi}{2L}\right)^2,\quad n=0,1,2,3,\ldots$	$\lambda_n=\left(\frac{(2n+1)\pi}{2L}\right)^2,\quad n=0,1,2,3,\ldots$
Eigenfunctions	$\phi_n(x)=\sin\left(\frac{(2n+1)\pi x}{2L}\right)$	$\phi_n(x)=\cos\left(\frac{(2n+1)\pi x}{2L}\right)$
Series Expansion	$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{(2n+1)\pi x}{2L}\right)$	$f(x)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{(2n+1)\pi x}{2L}\right)$
Coefficients	$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n+1)\pi x}{2L}\right)dx$	$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{(2n+1)\pi x}{2L}\right)dx$

Table 5.2: Boundary value problem for $\frac{d^2\phi}{dx^2}=-\lambda\,\phi$ with mixed boundary conditions