# Chapter 6

# The Wave Equation

In this chapter, we will explore the solution of the one-dimensional wave equation on a finite domain. The first approach is by using the method of separation of variables. The approach is quite similar to that used for the heat equation, but with key differences. In this case, the time equation becomes a second-order ordinary differential equation (ODE) with an indicial equation that yields complex roots. This results in time-dependent functions that are sines and cosines, rather than the exponential decay found in the heat equation.

Depending on the boundary conditions applied to the spatial ODE, we derive eigenvalue problems that are analogous to those encountered in the heat equation case. Each eigenfunction corresponds to a particular periodic extension of the solution. For instance, Dirichlet boundary conditions lead to eigenfunctions that are sine functions, which are associated with the odd periodic extension of the solution defined on the domain $(0, L)$.

We will also to derive the classical D'Alembert Solution to the wave equation on the domain $(-\infty, +\infty)$ with prescribed initial displacements and velocities. This solution fully describes the equations of motion of an infinite elastic string that has a prescribed shape and initial velocity.

We will demonstrate, using separation of variables, that the solution to the wave equation on a finite domain is essentially the D'Alembert solution. In this context, the initial condition functions are periodic extensions that match the boundary conditions specific to the problem at hand.

## 6.1 Solution of the Wave Equation by separation of variables

In this section we will apply the method of separation of variables to the one dimensional wave equation, given by

$$\begin{aligned} &\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0 \\ &\text{BC}: \quad u(0,t) = 0, \quad u(L,t) = 0 \\ &\text{IC}: \quad u(x,0) = f(x) \quad u_t(x,0) = g(x), \quad 0 < x < L \end{aligned} \tag{6.1}$$

This problem concerns wave propagation on a string of length $L$ with fixed ends, meaning the displacement at both ends is zero. The function $u(x,t)$ represents the vertical displacement of the string as a function of time. The wave equation is derived under the assumptions that the displacement remains small and the string is uniform. The wave speed $c$ is given by

$$c = \sqrt{\frac{\tau}{\mu}}$$

where $\tau$ represents the tension in the string and $\mu$ is the mass per unit length.
This principle is evident in string instruments. By adjusting the tension, different tones can be produced, while the properties of the string-such as material (e.g., nylon or steel) and thickness-also influence the sound. A thicker string increases mass density, which in turn lowers the frequency, explaining why piano bass strings produce deeper notes.

The term $u_{tt}$ represents the acceleration of a segment of the string, while $u_{xx}$ describes its concavity. A positive concavity indicates an upward curve, meaning adjacent points pull toward equilibrium. Conversely, a negative concavity leads to downward acceleration.

As usual, to apply separation of variables, we let $u(x,t) = X(x)T(t)$. Then find

$$X\ddot{T} = c^2 X'' T$$

which can be rewritten as

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}$$

Again, we have separated the functions of time on one side and space on the other side. Therefore, we set each function equal to a constant, $\lambda$.

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = \lambda$$

This leads to two equations:

$$\ddot{T} = c^2 \lambda T$$
$$X'' = \lambda X$$

As before, we have the boundary conditions on $X(x)$ :

$$X(0) = 0, \quad \text{and} \quad X(L) = 0$$

giving the solutions,

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = -\left(\frac{n\pi}{L}\right)^2, n = 1, 2, \ldots$$

The main difference from the solution of the heat equation is the form of the time equation which is a second-order ODE.

$$T'' + \left(\frac{n\pi c}{L}\right)^2 T = 0$$

The solutions to this latter equation are

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

The general solution, a superposition of all product solutions, is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \tag{6.2}$$

This solution satisfies the wave equation and the boundary conditions. We still need to satisfy the initial conditions. Note that there are two initial conditions, since the wave equation is second order in time. First, we have $u(x,0) = f(x)$. Thus,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \tag{6.3}$$

Hence,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{6.4}$$

In order to obtain the condition on the initial velocity, $u_t(x,0) = g(x)$, we need to differentiate the general solution with respect to $t$ :

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \tag{6.5}$$

Then, we have from the initial velocity

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right) \tag{6.6}$$

Hence,

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{6.7}$$

**Example 6.1.1.** *Solve the following Wave equation:*

$$\begin{array}{ll} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} & \text{for all } 0 < x < 1 \text{ and } t > 0 \\ u(0,t) = u(1,t) = 0 & \text{for all } t > 0 \\ u(x,0) = x(1-x) & \text{for all } 0 < x < 1 \\ u_t(x,0) = 0 & \text{for all } 0 < x < 1 \end{array}$$

*This is a special case of equation (6.1) with* $L = 1, f(x) = x(1-x)$ *and* $g(x) = 0$. *Therefore we have*

$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left[A_k \cos(ck\pi t) + B_k \sin(ck\pi t)\right]$$

*with*

$$A_k = 2\int_0^1 x(1-x)\sin(k\pi x)dx, \quad B_k = 0$$

*We have*

$$\int_0^1 x\sin(k\pi x)dx = -\cos(k\pi)\frac{1}{k\pi}$$
$$\int_0^1 x^2\sin(k\pi x)dx = \cos(k\pi)\frac{2-k^2\pi^2}{k^3\pi^3} - \frac{2}{k^3\pi^3}$$

*Hence,*

$$A_k = 2\int_0^1 x(1-x)\sin(k\pi x)dx = \frac{4}{k^3\pi^3}[1-\cos(k\pi)]$$
$$= \begin{cases} \frac{8}{k^3\pi^3} & \textit{for } k \textit{ odd} \\ 0 & \textit{for } k \textit{ even} \end{cases}$$

*and*

$$u(x,t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{8}{k^3\pi^3}\sin(k\pi x)\cos(ck\pi t)$$

*or*

$$\boxed{u(x,t) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3\pi^3}\sin((2k+1)\pi x)\cos(c(2k+1)\pi t)}$$

**Example 6.1.2.** *Now we consider*

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \textit{for all } 0 < x < 1 \textit{ and } t > 0$$
$$u(0,t) = u(1,t) = 0 \quad \textit{for all } t > 0$$
$$u(x,0) = \sin(5\pi x) + 2\sin(7\pi x) \quad \textit{for all } 0 < x < 1$$
$$u_t(x,0) = 0 \quad \textit{for all } 0 < x < 1$$

*This is again a special case of equations (6.1) with* $L = 1$*. So,*

$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left[A_k\cos(ck\pi t) + B_k\sin(ck\pi t)\right]$$

*This time it is very inefficient to use the integral formulae to evaluate* $A_k$ *and* $B_k$*. It is easier to observe directly, just by matching coefficients, that*

$$\sin(5\pi x) + 2\sin(7\pi x) = u(x,0) = \sum_{k=1}^{\infty} A_k\sin(k\pi x) \Rightarrow A_k = \begin{cases} 1 & \textit{if } k = 5 \\ 2 & \textit{if } k = 7 \\ 0 & \textit{if } k \neq 5, 7 \end{cases}$$

$$0 = u_t(x,0) = \sum_{k=1}^{\infty} ck\pi B_k\sin(k\pi x) \Rightarrow B_k = 0$$

$$\boxed{u(x,t) = \sin(5\pi x)\cos(5c\pi t) + 2\sin(7\pi x)\cos(7c\pi t)}$$

**Note!**

The following observations are in order.

- **Period and Frequency of Vibration**:

$$\cos\left(\frac{n\pi c}{L}(t+T)\right) = \cos\left(\frac{n\pi ct}{L}\right), \quad \text{provided} \quad \frac{n\pi cT}{L} = 2\pi$$

Thus, the period of mode $n$ is given by:

$$T_n = \left(\frac{2L}{c}\right)\frac{1}{n}$$

which represents the seconds per cycle. The natural frequencies of vibration are:

$$f_n = \frac{1}{T_n} = n\left(\frac{c}{2L}\right)$$

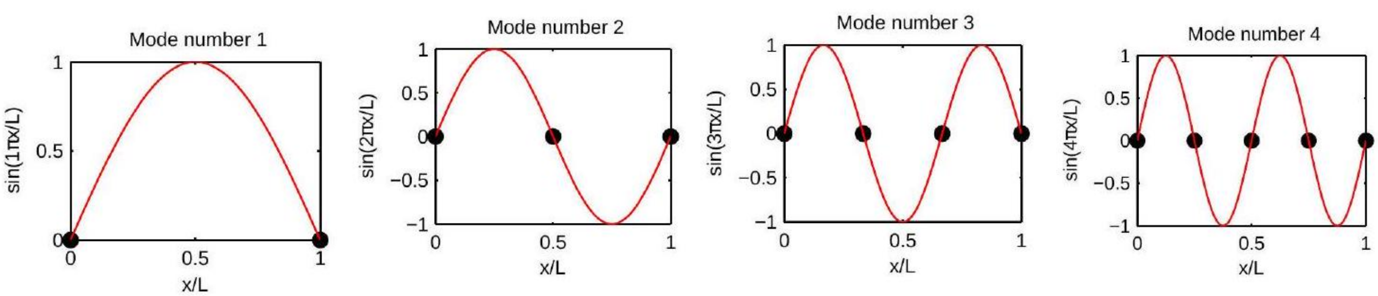
- **Modes of Vibration:** Standing waves have a wavelength given by:

$$\lambda_n = \frac{2L}{n}$$

In the following four figures, we plot the first four modes of vibration. The first, known as the fundamental mode of vibration, is associated with the lowest frequency:

$$f_1 = \frac{1}{T_1} = \left(\frac{c}{2L}\right)$$

All higher frequencies, also known as overtones, are integer multiples of this fundamental frequency. The nodes in these modal plots are indicated by solid circles, which represent the points at which the displacement associated with a given mode is zero.




(a) Fundamental mode (b) Second mode (c) Third mode (d) Fourth mode

$$X_1(x) = \sin\left(\frac{\pi x}{L}\right) \qquad X_2(x) = \sin\left(\frac{2\pi x}{L}\right) \qquad X_3(x) = \sin\left(\frac{3\pi x}{L}\right) \qquad X_4(x) = \sin\left(\frac{4\pi x}{L}\right)$$

## 6.2 Non-homogeneous Wave Equation

We consider the wave equation with a source:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + s(x,t) \\ \text{BCs}: \ & u(0,t) = u(L,t) = 0 \\ \text{ICs}: \ & u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{aligned}$$

To solve this, we first look for a particular solution $v(x,t)$ of the PDE and boundary conditions.

Then the general solution will be $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is the general solution of the homogeneous PDE $u_{tt} = c^2 u_{xx}$ and boundary conditions. To satisfy our initial conditions, we must take the initial conditions for $w$ as $w(x,0) = f(x) - v(x,0), w_t(x,0) = g(x) - v_t(x,0)$.

## Steady state

If the source term $s(x,t)$ does not depend on the time $t$ (so we can write $s(x,t) = s(x)$), then we can look for $v(x,t) = v(x)$ not depending on the time $t$. The PDE becomes $0 = c^2 v'' + s(x)$, and we must solve this subject to the boundary conditions $v(0) = v(L) = 0$. In this case it can be solved by integrating twice.

**Example 6.2.1.** *Consider the problem*

$$u_{tt} = u_{xx} + x$$

*BC:* $u(0,t) = u(1,t) = 0$
*IC:* $u(x,0) = 0, \quad u_t(x,0) = 1.$
*The differential equation says* $v'' = -x$. *One integration gives* $v' = -x^2/2 + A$ *where* $A$ *is a constant, another gives* $v = -x^3/6 + Ax + B$. *For* $v(0) = 0$ *we need* $B = 0$, *and then for* $v(1) = 0$ *we need* $-1/6 + A = 0$ *or* $A = 1/6$. *So* $v(x) = (x - x^3)/6$ *is our particular solution. The other part of the solution,* $w(x,t)$, *satisfies*

$$w_{tt} = w_{xx}$$

*boundary conditions* $w(0,t) = w(1,t) = 0$

$$\text{initial conditions } w(x,0) = -\left(x - x^3\right)/6, \quad w_t(x,0) = 1$$

*We could use a Fourier series for this:*

$$w(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right)$$

*where*

$$a_n = -\frac{1}{3}\int_0^1 \left(x - x^3\right)\sin(n\pi x)dx = \frac{2(-1)^n}{n^3\pi^3}$$
$$b_n = \frac{2}{n\pi}\int_0^1 \sin(n\pi x)dx = \frac{2\left(1 - (-1)^n\right)}{n^2\pi^2}$$

*And thus the complete solution is*

$$u(x,t) = \frac{x - x^3}{6} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^3\pi^3}\cos(n\pi t) + \frac{2\left(1 - (-1)^n\right)}{n^2\pi^2}\sin(n\pi t)\right)\sin(n\pi x)$$

## Exponential in $t$

If the source term is a function of $x$ times an exponential in $t$, we may look for a particular solution $v(x,t)$ that is also of this form. For example, consider

$$u_{tt} = u_{xx} + x\mathrm{e}^{-t}$$
$$\text{BCs} : u(0,t) = u(1,t) = 0$$
$$\text{ICs} : u(x,0) = 0, \quad u_t(x,0) = 1$$

We look for a solution of the form $v(x,t) = V(x)\mathrm{e}^{-t}$. Then the differential equation says $V(x)\mathrm{e}^{-t} = V''(x)\mathrm{e}^{-t} + x\mathrm{e}^{-t}$ or $V = V'' + x$. The general solution of this differential equation is $V(x) = x + c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x}$. The boundary conditions say $V(0) = 0 = c_1 + c_2$ and $V(1) = 0 = 1 + c_1\mathrm{e} + c_2\mathrm{e}^{-1}$. Solving for $c_1$ and $c_2$ we get $c_1 = -\mathrm{e}/\left(\mathrm{e}^2 - 1\right), c_2 = \mathrm{e}/\left(\mathrm{e}^2 - 1\right)$, i.e.

$$v(x,t) = \left(x - \frac{\mathrm{e}^{1+x}}{\mathrm{e}^2 - 1} + \frac{\mathrm{e}^{1-x}}{\mathrm{e}^2 - 1}\right)\mathrm{e}^{-t}$$

The initial conditions for $w(x,t)$ are

$$w(x,0) = -v(x,0) = -V(x) = -x + \frac{\mathrm{e}^{1+x} - \mathrm{e}^{1-x}}{\mathrm{e}^2 - 1}$$
$$w_t(x,0) = 1 - v_t(x,0) = 1 + V(x) = 1 + x - \frac{\mathrm{e}^{1+x} - \mathrm{e}^{1-x}}{\mathrm{e}^2 - 1}$$

## Arbitrary function of $x$ and $t$

We will use an eigenfunction expansion. We can write $u(x,t)$ and $s(x,t)$ for any $t$ as such a series, obtaining series expansions where the coefficients are functions of $t$ :

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$$
$$s(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

Our PDE will give us relations between these, which will be ordinary differential equations in $b_n(t)$ for each $n$.

**Example 6.2.2.** *Consider the problem*

$$u_{tt} = u_{xx} + xt$$
$$\text{BCs} : u(0,t) = u(1,t) = 0$$
$$\text{ICs} : u(x,0) = 0, \quad u_t(x,0) = 1$$

*The appropriate eigenfunctions for the homogeneous problem are* $\phi_n(x) = \sin(n\pi x)$*, the expansion being the Fourier sine series on the interval* $[0,1]$*. In particular,*

$$xt = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}t}{n\pi}\sin(n\pi x)$$

*so* $c_n(t) = 2(-1)^{n+1}t/(n\pi)$*. Putting these series into the differential equation, we get*

$$\sum_{n=1}^{\infty} b_n''(t)\sin(n\pi x) = -\sum_{n=1}^{\infty} b_n(t)(n\pi)^2\sin(n\pi x) + \sum_{n=1}^{\infty} c_n(t)\sin(n\pi x)$$

*By the uniqueness of Fourier series, the coefficients for each* $n$ *must match, i.e.*

$$b_n'' = -(n\pi)^2 b_n + c_n(t) = -(n\pi)^2 b_n - \frac{2(-1)^n t}{n\pi}$$

*The initial conditions for* $u$ *and* $u_t$ *give us initial conditions for* $b_n$ *and* $b_n'$ : $u(x,0) = 0$ *so* $b_n(0) = 0$, *and* $u_t(x,0) = 1 = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n}{n\pi} \sin(n\pi x)$ *so* $b_n'(0) = \frac{2(1-(-1)^n}{n\pi}$. *The general solution of the differential equation for* $b_n$ *is*

$$b_n(t) = A_n \cos(n\pi t) + B_n \sin(n\pi t) - \frac{2(-1)^n t}{(n\pi)^3}$$

*From the initial conditions we get* $b_n(0) = 0 = A_n$ *and* $b_n'(0) = \frac{2(1-(-1)^n}{n\pi} = n\pi B_n - \frac{2(-1)^n}{(n\pi)^3}$, *so* $B_n = \frac{2(-1)^n}{(n\pi)^4} + \frac{2(1-(-1)^n)}{(n\pi)^2}$. *The complete solution is*

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left( \left( \frac{2(-1)^n}{(n\pi)^4} + \frac{2\left(1-(-1)^n\right)}{(n\pi)^2} \right) \sin(n\pi t) - \frac{2(-1)^n t}{(n\pi)^3} \right) \sin(n\pi x)}$$

## 6.3 d'Alembert's solution of the Wave Equation

A general solution of the one-dimensional wave equation can be found. This solution was first derived by Jean-Baptiste le Rond d'Alembert (1717-1783) and is referred to as d'Alembert's formula. In this section, we derive d'Alembert's formula and then use it to obtain solutions to the wave equation on infinite, semi-infinite, and finite intervals.

We consider the wave equation in the form

$$u_{tt} = c^2 u_{xx} \tag{6.8}$$

and introduce the transformation

$$u(x,t) = U(\xi, \eta), \quad \text{where} \quad \xi = x + ct \quad \text{and} \quad \eta = x - ct. \tag{6.9}$$

Note that $\xi$ and $\eta$ are the characteristics of the wave equation. We also need to see how the derivatives transform. For example,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial U(\xi,\eta)}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ &= \frac{\partial U(\xi,\eta)}{\partial \xi} + \frac{\partial U(\xi,\eta)}{\partial \eta} \end{aligned}$$

Thus, as an operator, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

Similarly, one can show that

$$\frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$$

Using these results, the wave equation becomes

$$
\begin{aligned}
0 &= u_{tt} - c^2 u_{xx} \\
&= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u \\
&== \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} + c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} - c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right) U \\
&= -4c^2 \frac{\partial^2 U}{\partial \xi \partial \eta}
\end{aligned}
$$

This simplifies to

$$U_{\xi\eta} = 0$$

A further integration gives

$$U(\xi, \eta) = F(\xi) + G(\eta) \tag{6.10}$$

Thus, the general solution of the wave equation is

$$u(x,t) = F(x+ct) + G(x-ct), \tag{6.11}$$

where $F$ and $G$ are two arbitrary, twice differentiable functions. As $t$ increases, $F(x+ct)$ shifts to the left and $G(x-ct)$ shifts to the right, so that the solution may be seen as the sum of left- and right-traveling waves.

We now use initial conditions to determine the unknown functions. Suppose that

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad |x| < \infty \tag{6.12}$$

Then, applying these to the general solution, we obtain

$$f(x) = F(x) + G(x) \tag{6.13}$$

$$g(x) = c\left[F'(x) - G'(x)\right] \tag{6.14}$$

Integrating (6.14), we have

$$\frac{1}{c}\int_0^x g(s)dx = F(x) - G(x) - F(0) + G(0)$$

Adding this result to (6.13), gives

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)ds + \frac{1}{2}[F(0) - G(0)]$$

Subtracting from (6.13), gives

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds - \frac{1}{2}[F(0) - G(0)]$$

Now we can write out the solution $u(x,t) = F(x+ct) + G(x-ct)$, the d'Alembert solution:

$$\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds} \tag{6.15}$$

**Example 6.3.1.** *The d'Alembert solution of the wave equation:*

$$u_{tt} = 7u_{xx}$$
$$\text{BCs}: u(0,t) = u(1,t) = 0$$
$$\text{ICs}: u(x,0) = x^2, \quad u_t(x,0) = \cos(3x)$$

*is given by*

$$u(x,t) = \frac{(x+\sqrt{7}t)^2 + (x-\sqrt{7}t)^2}{2} + \frac{\sin(3(x+\sqrt{7}t)) - \sin(3(x-\sqrt{7}t))}{6\sqrt{7}}$$

**Example 6.3.2.** *We consider the following Wave equation:*

$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty \text{ and } t > 0$$
$$u(x,0) = f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
$$u_t(x,0) = 0$$

*Using d'Alembert's solution*

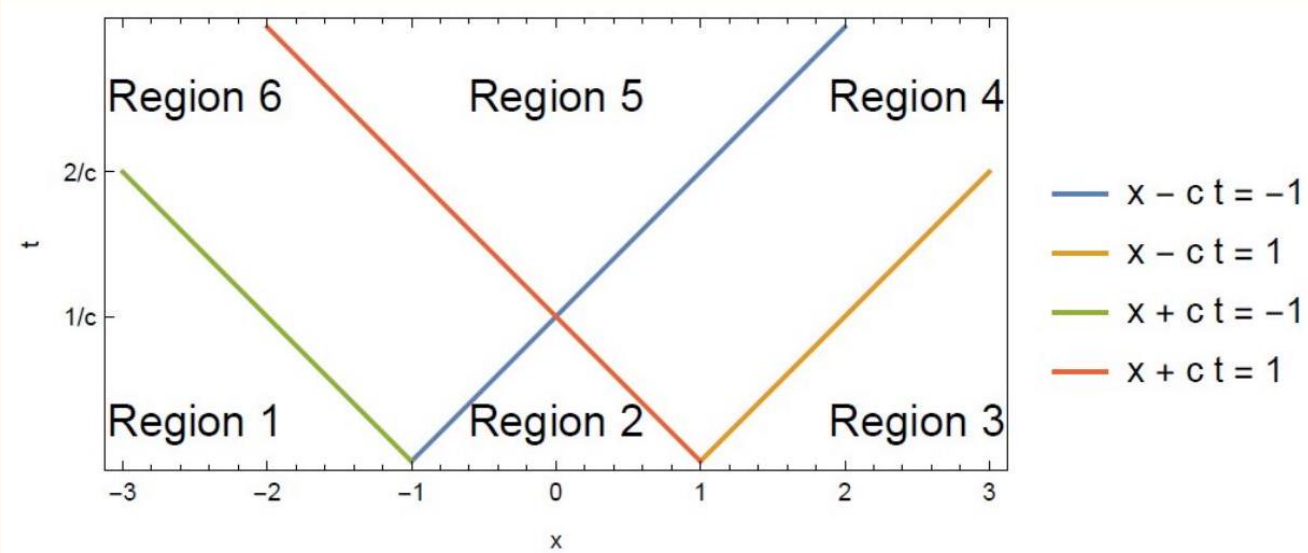
$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

*We note that:*

- *Along lines where $x + ct$ is constant the term $f(x + ct)$ is constant.*
- *Likewise along lines where $x - ct$ is constant the term $f(x - ct)$ is constant.*
- *These lines are called characteristics.*

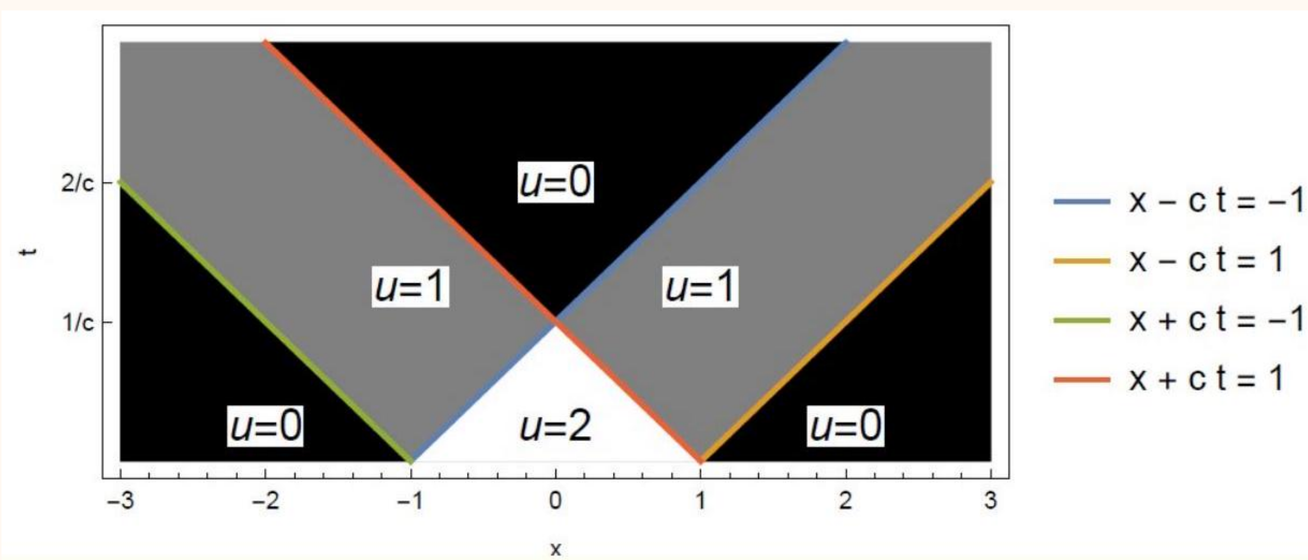
$$f(x) = \begin{cases} 2 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
$$\frac{1}{2}f(x+ct) = \begin{cases} 1 & \text{if } -1-ct < x < 1-ct \\ 0 & \text{otherwise} \end{cases}$$
$$\frac{1}{2}f(x-ct) = \begin{cases} 1 & \text{if } -1+ct < x < 1+ct \\ 0 & \text{otherwise} \end{cases}$$

*The characteristics where $x + ct = \pm 1$ and $x - ct = \pm 1$ help determine the solution.*




*Region 1:* $\{(x,t) \mid x+ct<-1\}$; *Region 2:* $\{(x,t) \mid -1<x-ct$ *and* $x+ct<1\}$ *Region 3:* $\{(x,t) \mid 1<x-ct\}$; *Region 4:* $\{(x,t) \mid 1<x+ct$ *and* $-1<x-ct<1\}$ *Region 5:* $\{(x,t) \mid 1<x+ct$ *and* $x-ct<-1\}$; *Region 6:* $\{(x,t) \mid -1<x+ct<1$ *and* $x-ct<-1\}$. *Hence,*

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)\\
&= \begin{cases}
0 & \textit{if } x+ct<-1\\
2 & \textit{if } -1<x-ct<1 \textit{ and } -1<x+ct<1\\
0 & \textit{if } 1<x-ct\\
1 & \textit{if } 1<x+ct \textit{ and } -1<x-ct<1\\
0 & \textit{if } 1<x+ct \textit{ and } x-ct<-1\\
1 & \textit{if } -1<x+ct<1 \textit{ and } x-ct<-1
\end{cases}
\end{aligned}
$$



## 6.4 Interpretation of the Fourier series solution in terms of d' Alembert's solution

Recall the double-angle trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

which we will use to interpret the Fourier series solution in terms of D'Alembert's solution for an infinite domain. Using the double angle formula, we obtain

$$\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left\{\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right\}$$
$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\left\{\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right\}$$

Now,

$$\begin{aligned}\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2}\sum_{n=1}^{\infty} A_n \left[\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right] \\ &= \frac{1}{2}\left[f_0(x+ct) + f_0(x-ct)\right]\end{aligned}$$

where $f_o$ is the odd periodic extension of $f$.
Similarly,

$$\begin{aligned}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left[\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)\right] \\ &= \frac{1}{2}[G(x-ct) - G(x+ct)] \qquad (6.16)\end{aligned}$$

where

$$G(x) := \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right), \quad B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

We can write

$$G(x) = \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left[\int_0^L g(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\cos\left(\frac{n\pi x}{L}\right)$$

Therefore,

$$\begin{aligned}G'(x) &= -\frac{1}{c}\sum_{n=1}^{\infty}\frac{2}{L}\left[\int_0^L g(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\sin\left(\frac{n\pi x}{L}\right) \\ &= -\frac{1}{c}g_o(x)\end{aligned}$$

Consequently,

$$G(x) = -\frac{1}{c}\int_0^x g_0(s)ds + D$$

where $D$ is an integration constant.
Returning to the series representation, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) &= \frac{1}{2}[G(x-ct)-G(x+ct)] \\
&= \frac{1}{2c}\left\{\left[-\int_0^{x-ct} g_o(s)ds + D\right] - \left[-\int_0^{x+ct} g_o(s)ds + D\right]\right\} \qquad (6.17) \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} g_o(s)ds \qquad (6.18)
\end{aligned}$$

Therefore, combining (6.13)) and (6.18) we obtain the following expression for the solution of the wave equation for a finite domain in the form of D'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left[f_o(x+ct) + f_o(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g_o(s)ds$$

where $f_o$ and $g_o$ are the odd periodic extensions of $f$ and $g$ on $[0, L]$, i.e.,

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0, \end{cases} \quad \text{with } f_o(x+2L) = f_o(x),$$

$$g_o(x) = \begin{cases} g(x), & 0 < x < L, \\ -g(-x), & -L < x < 0, \end{cases} \quad \text{with } g_o(x+2L) = g_o(x).$$

**Remark 6.4.1.** *In contrast to the Heat Equation, whose solutions typically decay exponentially over time, the solutions to the Wave Equation persist, as their time dependence involves* $\sin\left(\frac{n\pi ct}{L}\right)$ *and* $\cos\left(\frac{n\pi ct}{L}\right)$*, which do not decay over time.*