Chapter 7

The Laplace Equation

In this chapter, we begin our exploration of Laplace's equation, which describes the steady-state behavior of a field dependent on two or more independent variables, typically spatial. We illustrate how the inhomogeneous Dirichlet boundary value problem for the Laplacian on a rectangular domain can be decomposed into a series of four boundary value problems. Each of these problems features a single boundary segment with inhomogeneous boundary conditions, while the remaining boundaries satisfy homogeneous conditions. This approach allows us to solve each problem individually using the method of separation of variables.

Laplace's Equation arises as a steady state problem for the Heat or Wave Equations that do not vary with time so that $\frac{\partial u}{\partial t} = 0 = \frac{\partial^2 u}{\partial t^2}$. in 2 D , the equation reads;

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This equation is Laplace's equation in two dimensions. It is of the essential equations in applied mathematics (and the most important for time-independent problems). Note that in general, the Laplacian for a function $u(x_1, \cdots, x_n)$ in $\mathbb{R}^n \to \mathbb{R}$ is defined to be the sum of the second partial derivatives:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

The inhomogeneous case, i.e.

$$\Delta u = f,$$

the equation is called Poisson's equation. Innumerable physical systems are described by Laplace's equation or Poisson's equation, beyond steady states for the heat equation: inviscid fluid flow (e.g. flow past an airfoil), stress in a solid, electric fields, wavefunctions (time independence) in quantum mechanics, and more.

7.1 Rectangular domain

We can solve Laplace's equation in a bounded domain by the same techniques used for the heat and wave equation. In this section, we will solve Laplace's equation on a rectangle in $\mathbb{R}^2$. First, we consider the case of Dirichlet boundary conditions. That is, we consider the following boundary value problem. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < a,\ 0 < y < b\}$.

Dirichlet BCs

We want to look for a solution of the following,

$$\begin{aligned} & u_{xx} + u_{yy} = 0, \quad 0 < x < a,\ 0 < y < b, \\ \text{BCs :}\quad & u(0,y) = g_1(y), \quad u(a,y) = g_2(y), \quad 0 < y < b \\ & u(x,0) = f_1(x), \quad u(x,b) = f_2(x), \quad 0 < x < a. \end{aligned} \tag{7.1}$$

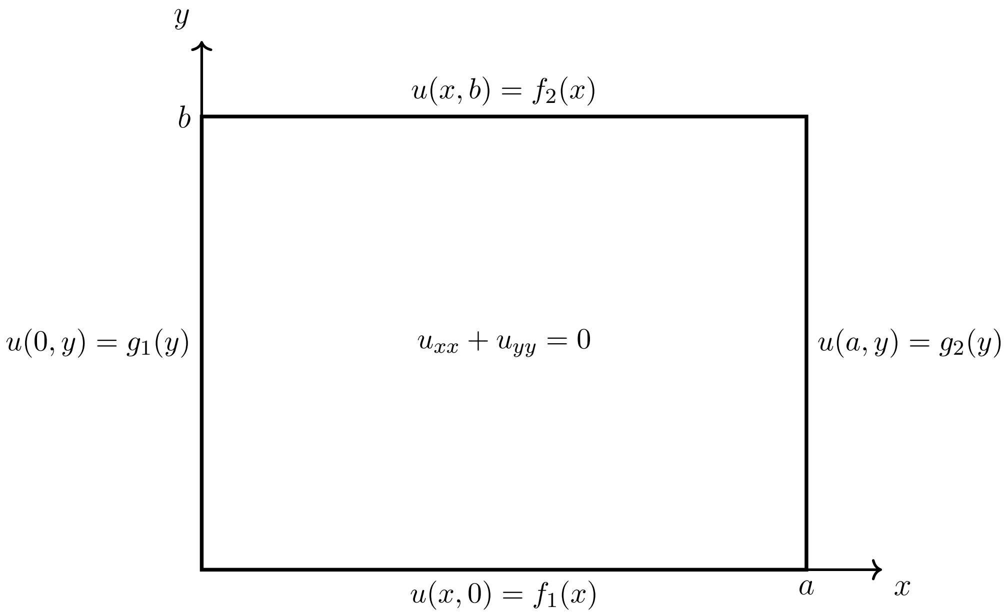

Figure 7.1: Laplace equation with boundary conditions on a rectangular domain.

The functions $f_1(x), f_2(x), g_1(y)$, and $g_2(y)$ are given functions of x and y, respectively. The partial differential equation is linear and homogeneous, but the boundary conditions, although linear, are not homogeneous. We cannot apply the method of separation of variables to this problem in its present form because, when we separate variables, the boundary value problem (which determines the separation constant) must have homogeneous boundary conditions. In this case, all the boundary conditions are nonhomogeneous.

Methodology 7.1.1. *We can address this difficulty by recognizing that the original problem is nonhomogeneous due to the four nonhomogeneous boundary conditions. The principle of superposition can sometimes be used for nonhomogeneous problems. To solve this, we decompose the problem into four subproblems, each having one nonhomogeneous condition. We define*

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

where each $u_i(x,y)$ satisfies Laplace's equation with one nonhomogeneous boundary condition and the related three homogeneous boundary conditions, as illustrated in Figure 7.2 Instead of directly solving for u, we will show how to solve for u_1, u_2, u_3, and u_4.

Methodology 7.1.2 (Solving for u_1). *Consider*

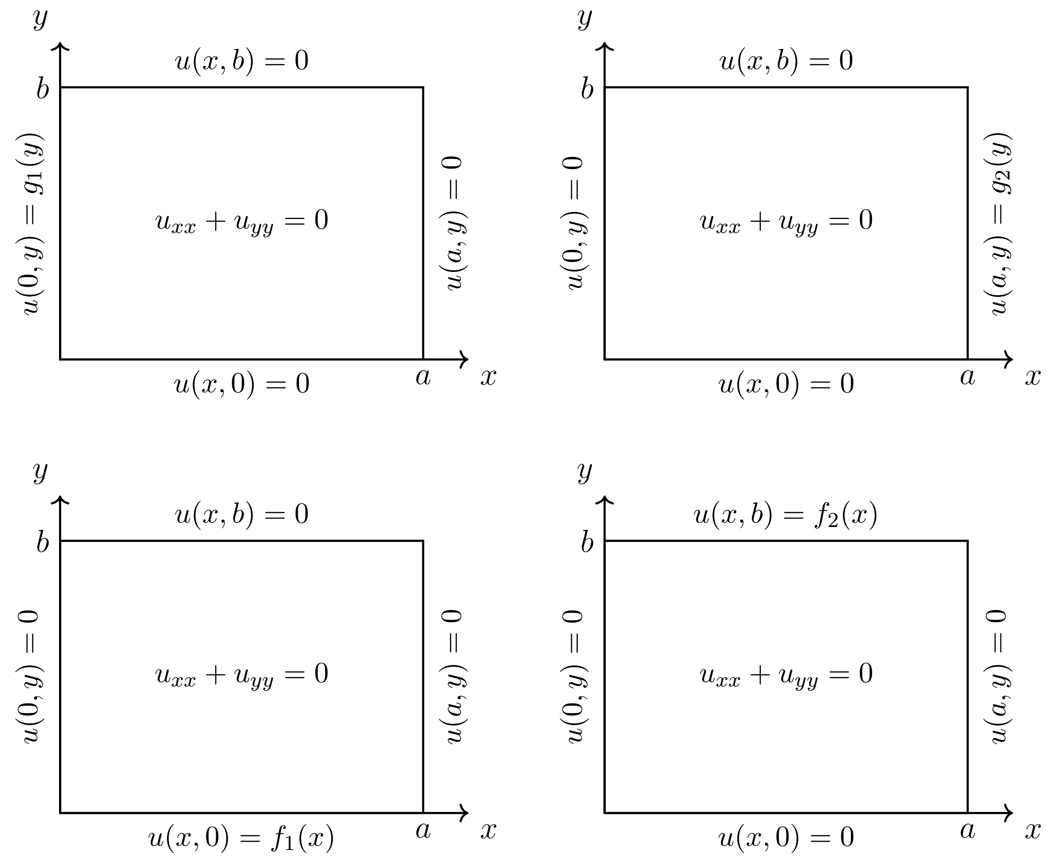

Figure 7.2: Decomposition of Laplace's equation with four subproblems, each having one nonhomogeneous boundary condition.

$$\begin{aligned} & u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \\ \text{BCs}: \ & u(0, y) = g_1(y) \quad u(a, y) = 0, \quad 0 < y < b \\ & u(x, 0) = 0, \quad u(x, b) = 0, \quad 0 < x < a. \end{aligned} \tag{7.2}$$

We use separation of variables. We look for a solution of the form

$$u(x, y) = X(x)Y(y).$$

Plugging this into our equation, we get

$$X''Y + XY'' = 0.$$

Now dividing by XY, we arrive at

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

which implies

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda$$

for some constant λ. By our boundary conditions, we want $Y(0) = 0 = Y(b)$. Therefore, we begin by solving the eigenvalue problem,

$$\begin{cases} Y'' = -\lambda Y & 0 < y < b \\ Y(0) = 0 = Y(b) \end{cases}$$

As we know, the solutions of this eigenvalue problem are given by

$$Y_n(y) = \sin\left(\frac{n\pi y}{b}\right), \lambda_n = \left(\frac{n\pi}{b}\right)^2$$

We now turn to solving

$$X'' = \left(\frac{n\pi}{b}\right)^2 X$$

with the boundary condition $X(a) = 0$. The solutions of this ODE are given by

$$X_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)$$

Now the boundary condition $X(a) = 0$ implies

$$A_n \cosh\left(\frac{n\pi a}{b}\right) + B_n \sinh\left(\frac{n\pi a}{b}\right) = 0$$

Therefore,

$$u_n(x,y) = X_n(x)Y_n(y) = \left[A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)\right] \sin\left(\frac{n\pi y}{b}\right)$$

where A_n, B_n satisfy the condition

$$A_n \cosh\left(\frac{n\pi a}{b}\right) + B_n \sinh\left(\frac{n\pi a}{b}\right) = 0$$

is a solution of Laplace's equation on Ω which satisfies the boundary conditions $u(x,0) = 0$, $u(x,b) = 0$, and $u(a,y) = 0$. As we know, Laplace's equation is linear. Therefore, we can take any combination of solutions $\{u_n\}$ and get a solution of Laplace's equation which satisfies these three boundary conditions. Therefore, we look for a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} \left[A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)\right] \sin\left(\frac{n\pi y}{b}\right)$$

where A_n, B_n satisfy

$$A_n \cosh\left(\frac{n\pi a}{b}\right) + B_n \sinh\left(\frac{n\pi a}{b}\right) = 0 \tag{7.3}$$

To solve our boundary-value problem (7.2), it remains to find coefficients A_n, B_n which satisfy the condition $u(0,y) = g_1(y)$. we need

$$u(0,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi y}{b}\right) = g_1(y)$$

That is, we want to be able to express g_1 in terms of its Fourier sine series on the interval $[0,b]$. We know that the coefficients A_n are given by

$$A_n = \frac{2}{b} \int_0^b g_1(y) \sin\left(\frac{n\pi y}{b}\right) dy$$

Substituting this value of A_n into (7.3), we deduce that

$$B_n = -\frac{2}{b}\coth\left(\frac{n\pi a}{b}\right)\int_0^b g_1(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

We have found a solution of (7.2) given by

$$\boxed{u_1(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty}\left[A_n\cosh\left(\frac{n\pi}{b}x\right) + B_n\sinh\left(\frac{n\pi}{b}x\right)\right]\sin\left(\frac{n\pi}{b}y\right)}$$

where A_n and B_n are given above.

Similarly, we find functions u_2, u_3 and u_4 which vanish on three of the sides but satisfy the fourth boundary condition.

Methodology 7.1.3 (Solving for u_2)**.** *Consider*

$$\begin{aligned} & u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \\ \text{BCs}: \quad & u(x,0) = 0, \quad u(x,b) = 0, \quad 0 < x < a, \\ & u(0,y) = 0, \quad u(a,y) = f(y), \quad 0 < y < b. \end{aligned} \tag{7.4}$$

We use separation of variables. We look for a solution of the form

$$u(x,y) = X(x)Y(y).$$

Plugging this into our equation, we get

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Now dividing by $X(x)Y(y)$ (assuming nonzero factors), we arrive at

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

which implies

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = -\lambda$$

for some constant λ. By our boundary conditions, we require $Y(0) = 0 = Y(b)$. Therefore, we begin by solving the eigenvalue problem

$$\begin{cases} Y''(y) = -\lambda Y(y), \quad 0 < y < b \\ Y(0) = 0, \quad Y(b) = 0 \end{cases}$$

As we know, nontrivial solutions exist only if

$$\lambda = \lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad n = 1, 2, \ldots,$$

with corresponding eigenfunctions

$$Y_n(y) = \sin\left(\frac{n\pi y}{b}\right)$$

We now turn to solving

$$X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0$$

with the boundary condition $X(0) = 0$ *(since* $u(0, y) = 0$ *). The solutions of this ODE are given by*

$$X_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)$$

The condition $X(0) = 0$ *forces*

$$X_n(0) = A_n = 0$$

so that

$$X_n(x) = B_n \sinh\left(\frac{n\pi x}{b}\right)$$

Therefore, a separated solution is given by

$$u_n(x, y) = \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Since Laplace's equation is linear, any linear combination of these solutions is also a solution. Hence, we look for a solution of the form

$$u(x, y) = \sum_{n=1}^{\infty} c_n u_n(x, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right)$$

At $x = a$*, we have*

$$u(a, y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi y}{b}\right) = f(y)$$

Thus, the coefficients c_n *are determined by expanding* $f(y)$ *in a Fourier sine series on the interval* $0 < y < b$ *:*

$$c_n = \frac{2}{b \sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy, \quad n = 1, 2, \ldots$$

This completes the solution of (7.4).

Neumann BCs

Let us consider an example with Neumann boundary conditions.

Example 7.1.1. *Consider*

$$\begin{aligned} & u_{xx} + u_{yy} = 0, \quad 0 < x < a, 0 < y < b, \\ \text{BCs}: \; & u_y(x, 0) = 0, \quad u_y(x, b) = 0, \quad 0 < x < a, \\ & u_x(0, y) = 0, \quad u_x(a, y) = f(y), \quad 0 < y < b. \end{aligned} \tag{7.5}$$

Assume a solution of the form

$$u(x,y) = X(x)Y(y)$$

Then,

$$u_{xx} = X''(x)Y(y) \quad \text{and} \quad u_{yy} = X(x)Y''(y)$$

Substituting into Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assuming both factors are nonzero) yields

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

Since the left-hand side is a sum of a function of x and a function of y, we set

$$\frac{X''(x)}{X(x)} = \lambda, \quad \frac{Y''(y)}{Y(y)} = -\lambda$$

for some constant λ.

The Y equation becomes

$$Y''(y) + \lambda Y(y) = 0$$

with the boundary conditions (implied by $u_y(x,0) = 0$ and $u_y(x,b) = 0$):

$$Y'(0) = 0, \quad Y'(b) = 0$$

Nontrivial solutions exist only if

$$\lambda = \lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad n = 0, 1, 2, \ldots$$

with corresponding eigenfunctions

$$Y_n(y) = \cos\left(\frac{n\pi y}{b}\right)$$

With $\lambda = \left(\frac{n\pi}{b}\right)^2$, the X equation is

$$X''(x) - \left(\frac{n\pi}{b}\right)^2 X(x) = 0$$

with the boundary condition (from $u_x(0,y) = 0$):

$$X'(0) = 0$$

The general solution of the ODE is

$$X_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right) + B_n \sinh\left(\frac{n\pi x}{b}\right)$$

The condition $X'(0) = 0$ forces

$$X_n'(0) = A_n \frac{n\pi}{b} \sinh(0) + B_n \frac{n\pi}{b} \cosh(0) = B_n \frac{n\pi}{b} = 0$$

so that

$$B_n = 0$$

Thus,

$$X_n(x) = A_n \cosh\left(\frac{n\pi x}{b}\right)$$

Therefore, a separated solution is given by

$$u_n(x,y) = X_n(x)Y_n(y) = \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Notice that for $n = 0$ the corresponding solution is constant in y.
Since Laplace's equation is linear, the general solution is

$$u(x,y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)$$

At $x = a$, differentiating with respect to x yields

$$u_x(x,y) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{b} \sinh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)$$

so that

$$u_x(a,y) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right)\cos\left(\frac{n\pi y}{b}\right) = f(y)$$

Therefore, we must choose the constants $c_1, c_2, \ldots$ such that

$$f(y) = \sum_{n=1}^{\infty} \frac{n\pi}{b} c_n \sinh\left(\frac{n\pi a}{b}\right)\cos\left(\frac{n\pi y}{b}\right), \quad 0 < y < b \tag{7.6}$$

Now, we know that we can expand $f(y)$ in the cosine series

$$f(y) = \frac{1}{b}\int_0^b f(y)dy + \frac{2}{b}\sum_{n=1}^{\infty}\left[\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)dy\right]\cos\left(\frac{n\pi y}{b}\right) \tag{7.7}$$

on the interval $0 \leq y \leq b$. However, we cannot equate coefficients in (7.6) and (7.7) since the series (7.6) has no constant term. Therefore, the condition

$$\int_0^b f(y)dy = 0$$

is necessary for this Neumann problem to have a solution. In that case,

$$c_n = \frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)dy, \quad n \geq 1$$

Note that the coefficient c_0 remains arbitrary. In fact, the solution $u(x,y)$ is determined only up to an additive constant, which is a characteristic feature of Neumann problems.

Mixed BCs

We now consider an example where we have a mixed boundary condition on one side.

Example 7.1.2. *Consider*

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad (x,y) \in \Omega, \quad \Omega = \{(x,y) : 0 < x < a, 0 < y < b\}, \\ \text{BCs}: \ &u(0,y) = 0, \quad u(a,y) = 0, \quad 0 < y < b, \\ &u(x,0) - u_y(x,0) = 0, \quad u(x,b) = f(x), \quad 0 < x < a. \end{aligned}$$

We use separation of variables by seeking a solution of the form

$$u(x,y) = X(x)Y(y).$$

Substituting into Laplace's equation, we have

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assuming neither is zero) yields

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

so that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

for some separation constant λ. We first solve the X-problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < a, \quad X(0) = 0, \quad X(a) = 0.$$

It is well known that the eigenfunctions and eigenvalues are

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, \ldots$$

For each n with $\lambda = \lambda_n$, the Y-equation becomes

$$Y''(y) = \left(\frac{n\pi}{a}\right)^2 Y(y)$$

Its general solution is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right)$$

The boundary condition at $y = 0$ is

$$u(x,0) - u_y(x,0) = 0$$

Since $u(x,y) = X_n(x)Y_n(y)$ and $X_n(x) \not\equiv 0$, we require

$$Y_n(0) - Y_n'(0) = 0$$

Noting that

$$Y_n(0) = A_n \quad \text{and} \quad Y_n'(0) = B_n \frac{n\pi}{a}$$

this condition yields

$$A_n - B_n \frac{n\pi}{a} = 0, \quad \text{or} \quad A_n = \frac{n\pi}{a} B_n$$

Thus, we may write

$$Y_n(y) = B_n \left[\frac{n\pi}{a}\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)\right]$$

The separated solution corresponding to the n-th term is therefore

$$u_n(x,y) = X_n(x)Y_n(y) = B_n \sin\left(\frac{n\pi x}{a}\right)\left[\frac{n\pi}{a}\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)\right]$$

Since Laplace's equation is linear, we may superimpose these solutions and write

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\left[\frac{n\pi}{a}\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)\right]$$

To satisfy the boundary condition $u(x,b) = f(x)$, we substitute $y = b$ into the series:

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\left[\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right) + \sinh\left(\frac{n\pi b}{a}\right)\right] = f(x)$$

Express $f(x)$ in its Fourier sine series on $(0,a)$:

$$f(x) \sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

with coefficients

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

Thus, equating the coefficients of like sine functions, we require

$$B_n \left[\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right) + \sinh\left(\frac{n\pi b}{a}\right)\right] = A_n$$

That is,

$$B_n = \frac{2}{a}\left[\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right) + \sinh\left(\frac{n\pi b}{a}\right)\right]^{-1}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

In summary, the solution of the boundary value problem is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\left[\frac{n\pi}{a}\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)\right]$$

where

$$B_n = \frac{2}{a}\left[\frac{n\pi}{a}\cosh\left(\frac{n\pi b}{a}\right) + \sinh\left(\frac{n\pi b}{a}\right)\right]^{-1}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

7.2 Laplace's Equation on a disk

In this section, we consider Laplace's Equation on a disk in $\mathbb{R}^2$. That is, let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ u = h(\theta) & (x,y) \in \partial\Omega \end{cases}$$

To solve, we write this equation in polar coordinates as follows. To transform our equation in to polar coordinates, we will write the operators ∂_x and ∂_y in polar coordinates. We use the fact that

$$\begin{aligned} x^2 + y^2 &= r^2 \\ \frac{y}{x} &= \tan\theta \end{aligned}$$

Consider a function u such that $u = u(r,\theta)$, where $r = r(x,y)$ and $\theta = \theta(x,y)$. That is,

$$u = u(r(x,y), \theta(x,y))$$

Then

$$\begin{aligned} \frac{\partial}{\partial x} u(r(x,y), \theta(x,y)) &= u_r r_x + u_\theta \theta_x \\ &= u_r \frac{x}{(x^2+y^2)^{1/2}} - u_\theta \frac{y}{x^2 \sec^2\theta} \\ &= u_r \cos\theta - \frac{\sin\theta}{r} u_\theta \end{aligned}$$

Therefore, the operator $\frac{\partial}{\partial x}$ can be written in polar coordinates as

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

Similarly, the operator $\frac{\partial}{\partial y}$ can be written in polar coordinates as

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

Now squaring these operators we have

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right]^2 \\ &= \cos^2\theta \frac{\partial^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= \left[\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right]^2 \\ &= \sin^2\theta \frac{\partial^2}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} \end{aligned}$$

Combining the above terms, we can write the operator $\partial_x^2 + \partial_y^2$ in polar coordinates as follows,

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Methodology 7.2.1 (Solving Laplace's equation in polar coordinates)**.** *Let the Laplace equation in polar coordinates be*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < a, \quad 0 \leq \theta \leq 2\pi$$

with the boundary condition

$$u(a,\theta) = h(\theta), \quad 0 \leq \theta \leq 2\pi$$

There are no explicit boundary conditions in θ; however, because θ is an angle there are implied periodic boundary conditions

$$u(r,0) = u(r,2\pi), \quad u_\theta(r,0) = u_\theta(r,2\pi)$$

We look for a solution by separation of variables, assuming

$$u(r,\theta) = R(r)\Theta(\theta)$$

Substituting into the equation yields

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

Dividing by $R(r)\Theta(\theta)$ (with $R, \Theta \not\equiv 0$) gives

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

Multiplying through by r^2 leads to

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

Since the first two terms depend only on r and the last only on θ, we set

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\left(r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)}\right) = \lambda$$

where λ is a separation constant.
Thus, the Θ-equation becomes

$$\Theta''(\theta) - \lambda\Theta(\theta) = 0$$

with the periodic conditions $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. These imply that

$$\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad \lambda_n = -n^2, \quad n = 0, 1, 2, \ldots$$

For each n, the corresponding R-equation is

$$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = 0$$

Assuming a solution of the form $R(r) = r^\alpha$ leads to the indicial equation

$$\alpha^2 - n^2 = 0$$

Thus, we have two distinct solutions $\alpha = \pm n$ when n is positive, and one double root $\alpha = 0$ when n is zero. Correspondingly, we get the general solution for the radial Euler's equation

$$\begin{aligned} R_n(r) &= C_n r^n + D_n r^{-n} \quad \text{for } n = 1, 2, 3, \ldots, \\ R_0(r) &= C_0 + D_0 \ln r \qquad \text{for } n = 0 \end{aligned}$$

Since the radial function R must be bounded at the origin, we are forced to set all values of D 's vanish, and the general solution becomes

$$R_n(r) = r^n, \quad n \geq 0$$

Thus, the separated solutions are

$$u_n(r,\theta) = r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

By linearity, the general solution is given by

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

To satisfy the boundary condition $u(a,\theta) = h(\theta)$, we require

$$\sum_{n=0}^{\infty} a^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] = h(\theta)$$

Using the orthogonality of the trigonometric functions on $[0, 2\pi]$, the Fourier coefficients are determined by

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi, \quad A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)d\phi, \quad n = 1, 2, \ldots$$
$$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)d\phi, \quad n = 1, 2, \ldots$$

Hence, the solution in series form is

$$\begin{aligned} u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)d\phi + \sum_{n=1}^{\infty} r^n \Bigg[\frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\cos(n\phi)d\phi\cos(n\theta) \\ + \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi)\sin(n\phi)d\phi\sin(n\theta)\Bigg] \end{aligned}$$

This series can be rewritten as a single integral by noting that

$$\cos(n(\theta - \phi)) = \cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)$$

Thus,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta - \phi))\right\}d\phi$$

Recognizing the sum as a geometric series, we have

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi)) = \frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}$$

Therefore, the solution is given in closed form by

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\phi)\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi$$

Example 7.2.1. *Consider the interior Dirichlet problem*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r < 2, \quad u(2,\theta) = f(\theta) \equiv \begin{cases} 1, & \text{if } 0 \le \theta \le \pi \\ \cos^2\theta, & \text{if } \pi \le \theta \le 2\pi \end{cases}$$

Its solution is known to be

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{2}\right)^n \left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi$$

To satisfy the boundary condition $u(2,\theta) = f(\theta)$, we calculate the coefficients as usual

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\phi + \frac{1}{\pi}\int_0^{\pi}\cos^2\phi \ \mathrm{d}\phi = \frac{3}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi}\cos(n\phi)\mathrm{d}\phi + \frac{1}{\pi}\int_0^{\pi}\cos(n\phi)\cos^2\phi \ \mathrm{d}\phi = \begin{cases} \frac{1}{4}, & \text{if } n = 2 \\ 0, & \text{if } n \ne 2 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin(n\phi)\mathrm{d}\phi + \frac{1}{\pi}\int_0^{\pi}\sin(n\phi)\cos^2\phi \ \mathrm{d}\phi = \begin{cases} \frac{-4}{n(n^2-4)}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Therefore, the required solution is

$$u(r,\theta) = \frac{3}{4} + \frac{r^2}{16}\cos(2\theta) - \frac{4}{\pi}\sum_{k\ge 0}\frac{1}{(2k+1)\left[(2k+1)^2-4\right]}\left(\frac{r}{2}\right)^{2k+1}\sin[(2k+1)\theta]$$

Example 7.2.2. *Consider the interior Dirichlet problem*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r < 3, \quad u(3,\theta) = 2\sin 4\theta - 3\cos 7\theta$$

Its solution is known to be

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{3}\right)^n \left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

To satisfy the boundary condition $u(2,\theta) = f(\theta)$, we have the expansion

$$u(3,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

where f is a combination of eigenfunctions. So we know that all coefficients in the above expansion are zeroes except $n = 4$ and $n = 7$. Hence,

$$b_4 = 2 \quad \textit{and} \quad a_7 = -3$$

This yields

$$u(r,\theta) = 2\left(\frac{r}{3}\right)^4 \sin(4\theta) - 3\left(\frac{r}{3}\right)^7 \sin(7\theta)$$

Methodology 7.2.2 (Solving Laplace's equation with Neumann BCs inside the circle). *Consider the interior Neumann problem for a circle of radius a :*

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \le r < a, \quad \left.\frac{\partial u}{\partial r}\right|_{r=a} = g(\theta), \quad \int_0^{2\pi} g(\theta)\mathrm{d}\theta = 0$$

where g is a given function. The general solution of Laplace's equation inside a circle of radius a is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le r < a, \quad 0 \le \theta \le 2\pi$$

Its derivative with respect to r becomes (assuming uniform convergence of the above series)

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty}\frac{n}{r}\left(\frac{r}{a}\right)^n \left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le r < a, \quad 0 \le \theta \le 2\pi$$

Setting r equals to a yields

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = g(\theta) = \sum_{n=1}^{\infty}\frac{n}{a}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le \theta \le 2\pi$$

The coefficients of the Fourier series are obtained by:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} g(\phi)\mathrm{d}\phi = 0$$
$$a_n = \frac{a}{n\pi}\int_0^{2\pi} g(\phi)\cos(n\phi)\mathrm{d}\phi, \quad n = 1, 2, \ldots$$
$$b_n = \frac{a}{n\pi}\int_0^{2\pi} g(\phi)\sin(n\phi)\mathrm{d}\phi, \quad n = 1, 2, \ldots$$

Note that the coefficient a_0 must be zero because the corresponding Fourier series for

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = \sum_{n=1}^{\infty}\frac{n}{a}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right], \quad 0 \le \theta \le 2\pi$$

does not contain a free term. Therefore, a Neumann problem has a solution if and only if the integral over the boundary vanishes:

$$\int_0^{2\pi} g(\phi)\mathrm{d}\phi = 0$$

Then the general solution to a Neumann problem is not unique but up to an arbitrary additive constant:

$$
\begin{aligned}
u(r,\theta) &= C + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \\
&= C + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \left[\frac{a}{n\pi}\int_0^{2\pi} g(\phi)\cos(n\phi)\mathrm{d}\phi \cos(n\theta) + \frac{a}{n\pi}\int_0^{2\pi} g(\phi)\sin(n\phi)\mathrm{d}\phi \sin(n\theta)\right]
\end{aligned}
$$

7.3 Semi-infinite strip problems

Example 7.3.1 (Homogeneous Bcs). *Consider the Laplace equation:*

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < \infty \tag{7.8}$$

with the boundary conditions:

$$u(0,y) = 0, \quad u(a,y) = 0 \tag{7.9a}$$

$$u(x,0) = f(x), \quad u(x,y) \to 0 \text{ as } y \to \infty \tag{7.9b}$$

We use the method of separation of variables and assume a solution of the form:

$$u(x,y) = X(x)Y(y)$$

Substituting into (7.8), we obtain:

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Dividing by $X(x)Y(y)$ (assuming nonzero factors), we separate the variables:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Therefore,

$$\frac{X''(x)}{X(x)} = \lambda, \quad -\frac{Y''(y)}{Y(y)} = \lambda \tag{7.10}$$

From (7.10), the equation for $X(x)$ is:

$$X''(x) - \lambda X(x) = 0$$

Using the boundary conditions (7.9a), we solve the eigenvalue problem with BC:

$$X(0) = 0, \quad X(a) = 0$$

Nontrivial solutions exist only if

$$\lambda_n = -\left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions:

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

The equation for $Y(y)$ from (7.10) is:

$$Y''(y) - \left(\frac{n\pi}{a}\right)^2 Y(y) = 0$$

The general solution is:

$$Y(y) = A_n e^{-\frac{n\pi}{a}y} + B_n e^{\frac{n\pi}{a}y}$$

Since $u(x,y) \to 0$ as $y \to \infty$, we must set $B_n = 0$, giving:

$$Y_n(y) = A_n e^{-\frac{n\pi}{a}y}$$

The general solution is given by:

$$u(x,y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi x}{a}\right)$$

Using the boundary condition at $y = 0$ from (7.9b), we require:

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) = f(x)$$

This represents the Fourier sine series of $f(x)$ on $0 < x < a$, giving:

$$c_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) dx$$

Thus, the final solution is:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{a}\left[\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) dx\right] e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi x}{a}\right)$$

Example 7.3.2 (Nonhomogeneous BCs). *Consider the Laplace equation:*

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < \infty.$$

with the boundary conditions:

$$u(0,y) = A, \quad u(a,y) = B$$
$$u(x,0) = f(x), \quad u(x,y) \to 0 \text{ as } y \to \infty$$

We seek a function $v(x)$ such that $v''(x) = 0$ and it satisfies the inhomogeneous boundary conditions:

$$v(0) = A, \quad v(a) = B.$$

The general solution for $v(x)$ is:

$$v(x) = \alpha x + \beta.$$

Using the boundary conditions:

$$A = v(0) = \beta$$
$$B = v(a) = \alpha a + A$$

we solve for α *and* β *:*

$$\alpha = \frac{B-A}{a}, \quad \beta = A.$$

Thus,

$$v(x) = \left(\frac{B-A}{a}\right)x + A.$$

Define $w(x,y)$ *such that:*

$$u(x,y) = v(x) + w(x,y).$$

Substituting into the PDE, we obtain:

$$w_{xx} + w_{yy} = 0.$$

The boundary conditions for $w(x,y)$ *are:*

$$w(0,y) = 0, \quad w(a,y) = 0$$
$$w(x,0) = f(x) - v(x)$$

Since w *satisfies the same boundary value problem as in the previous example, we directly obtain the solution:*

$$w(x,y) = \sum_{n=1}^{\infty} d_n e^{-\left(\frac{n\pi}{a}\right)y} \sin\left(\frac{n\pi x}{a}\right),$$

where the coefficients d_n *are given by:*

$$d_n = \frac{2}{a}\int_0^a [f(x) - v(x)] \sin\left(\frac{n\pi x}{a}\right) dx.$$

The general solution is therefore given by;

$$\boxed{u(x,y) = (B-A)\frac{x}{a} + A + \sum_{n=1}^{\infty} d_n e^{-\left(\frac{n\pi}{a}\right)y} \sin\left(\frac{n\pi x}{a}\right).}$$

Remark 7.3.1 (Inhomogeneous Laplace's equation). *Just as with the heat equation, if there are more complicated inhomogeneous terms, e.g.*

$$0 = u_{xx} + u_{yy} + f(x,y)$$

then the eigenfunction method is required unless you are lucky and there is a "particular" solution you can subtract out to remove the inhomogeneous terms.
When applying the eigenfunction method, one must pick a direction for the eigenfunctions, either

$$u = \sum_n c_n(x)\phi_n(y) \quad \text{or} \quad u = \sum_n c_n(y)\phi_n(x)$$

The correct choice is one where the boundary conditions are homogeneous (if both work, then it does not matter which you choose). The details are somewhat involved but straightforward in concept.

Note!

More examples of problems related to the Laplace equation can be found in Lecture Notes 27 by Prof. Peirce at Prof. Peirce's lectures