Chapter 8

Sturm-Liouville boundary value problems

In this chapter, we consider a class of two-point boundary value problems. The so-called Sturm-Liouville Problems. They are a class of eigenvalue problems, which include many of the previous problems as special cases.

We let

$$\mathcal{L}\phi = -\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi$$

Suppose we have some Sturm-Liouville problem with differential equation

$$\mathcal{L}\phi = \lambda r\phi \quad \text{for} \quad 0 < x < l \tag{8.1}$$

Here, the functions p, p', q, and r are continuous on $[0, \ell]$ with

$$p(x) \geq 0 \quad \text{and} \quad r(x) > 0, \quad 0 \leq x \leq \ell$$

We couple (8.1), with the following boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \quad \beta_1 y(\ell) + \beta_2 y'(\ell) = 0 \tag{8.2}$$

We define the Sturm-Liouville eigenvalue problem (SL Problem) as:

$$\begin{gathered} \mathcal{L}y = \lambda r(x)y \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(\ell) + \beta_2 y'(\ell) = 0 \\ p(x) > 0, \quad r(x) > 0 \end{gathered} \tag{8.3}$$

Remark 8.0.1.

- *If in* (8.3) *we choose* $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 1$ *and* $\beta_2 = 0$, *then we obtain the following problem*

$$\left.\begin{array}{c} y'' + \lambda y = 0 \\ y(0) = 0 = y(\ell) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 1, 2, \ldots \\ y_n(x) = \sin\left(\frac{n\pi x}{\ell}\right) \end{array}\right.$$

- *If in* (8.3) *we choose* $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 0$ *and* $\beta_2 = 1$, *then we obtain the following problem*

$$\left.\begin{array}{c} y'' + \lambda y = 0 \\ y'(0) = 0 = y'(\ell) \end{array}\right\} \Longrightarrow \left\{ \begin{array}{c} \lambda_n = \left(\frac{n\pi}{\ell}\right)^2, \quad n = 0, 1, 2, \ldots \\ y_n(x) = \cos\left(\frac{n\pi x}{\ell}\right) \end{array}\right.$$

- *Notice that these boundary conditions are specified at separate endpoints and are called separated boundary conditions. The periodic BC* $X(-\ell) = X(\ell)$ *are not separated, so the following problem is not technically an SL Problem.*

$$y'' + \lambda y = 0$$
$$y(-\ell) = y(\ell)$$
$$y'(-\ell) = y'(\ell)$$

- *If* $p > 0$ *and* $r > 0$ *on a finite interval* $[0, \ell]$*, then the SL problem is said to be regular. If either* $p(x)$ *or* $r(x)$ *is zero for some* x*, or if the domain is unbounded (e.g.,* $[0, \infty)$ *), then the problem is singular.*

- *If* P_0*,* P_1*,* P_2*, and* R *are continuous and* P_0 *and* R *are positive on a closed interval* $[a, b]$*, then the general equation*

$$P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0$$

can be rewritten in the SL form. Indeed, if we divide the previous equation by $P_0(x)$*, we obtain*

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y + \lambda\frac{R(x)}{P_0(x)}y = 0$$

We multiply the equation by the integrating factor $\tilde{p}(x) = e^{\int \frac{P_1(x)}{P_0(x)}dx}$*, we get:*

$$\tilde{p}(x)y'' + \tilde{p}(x)(x)\frac{P_1(x)}{P_0(x)}y' + \tilde{p}(x)\left(\frac{P_2(x)}{P_0(x)} + \lambda\frac{R(x)}{P_0(x)}\right)y = 0$$

The first two terms give:

$$\frac{d}{dx}\left(\tilde{p}(x)\frac{dy}{dx}\right)$$

Thus, the equation becomes:

$$\frac{d}{dx}\left(\tilde{p}(x)\frac{dy}{dx}\right) + \tilde{p}(x)\frac{P_2(x)}{P_0(x)}y = -\lambda\tilde{p}(x)\frac{R(x)}{P_0(x)}y$$

Define the following functions:

$$p(x) = -\tilde{p}(x) = -e^{\int \frac{P_1(x)}{P_0(x)}dx}$$
$$q(x) = \tilde{p}(x)\frac{P_2(x)}{P_0(x)}$$
$$r(x) = -\tilde{p}(x)\frac{R(x)}{P_0(x)}$$

Therefore,

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda r(x)y$$

Example 8.0.1.

1. *Consider the boundary value problem*

$$\phi'' + x\phi' + \lambda\phi = 0, \quad 0 < x < 1$$
$$\phi(0) = 0, \quad \phi(1) = 0$$

To write it into SL form, multiply by the integrating factor

$$\mu(x) = e^{\int x dx} = e^{x^2/2}$$

Since $P(x) = 1$, $Q(x) = x$ and $R(x) = 1$, we have:

$$e^{x^2/2}\phi'' + e^{x^2/2}x\phi' + \lambda e^{x^2/2}\phi = 0$$

That is

$$-\left(e^{x^2/2}\phi'\right)' = \lambda e^{x^2/2}\phi$$

Thus, the SL form is obtained with

$$p(x) = e^{x^2/2} \quad \text{and} \quad r(x) = e^{x^2/2}$$

2. *Consider the boundary value problem*

$$-y'' + x^4 y' = \lambda y$$

To convert into the SL form, we rewrite the equation as:

$$y'' - x^4 y' = -\lambda y$$

The integrating factor is

$$\mu(x) = e^{-\int x^4 dx} = e^{-x^5/5}$$

Multiplying the original equation by $\mu(x)$, we obtain:

$$-e^{-x^5/5}y'' + e^{-x^5/5}x^4 y' = \lambda e^{-x^5/5}y$$

That is

$$-\left(e^{-x^5/5}y'\right)' = \lambda e^{-x^5/5}y$$

Thus, the SL form is obtained with

$$p(x) = e^{-x^5/5} \quad \text{and} \quad r(x) = e^{-x^5/5}$$

3. *Consider the eigenvalue problem*

$$y'' + 3y' + (2 + \lambda)y = 0, \quad y(0) = 0, \quad y(1) = 0$$

The integrating factor is

$$\mu(x) = e^{\int 3dx} = e^{3x}$$

Multiplying the equation by $\mu(x)$ yields

$$-\left(-e^{3x}y'\right)' + 2e^{3x}y + \lambda e^{3x}y = 0$$

So that the eigenvalue problem can be written as

$$-\left(-e^{3x}y'\right)' + 2e^{3x}y = -\lambda e^{3x}y, \quad y(0) = 0, \quad y(1) = 0$$

Thus the SL form is obtained with

$$p(x) = -e^{3x}, \quad q(x) = 2e^{3x}, \quad \text{and} \quad r(x) = -e^{3x}$$

8.1 Properties of SL problems

Eigenvalues:

(a) The eigenvalues λ are all real.
(b) There are infinitely many eigenvalues λ_j satisfying

$$\lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty \qquad \text{as } j \to \infty$$

(c) Provided $\frac{\alpha_1}{\alpha_2} < 0$, $\frac{\beta_1}{\beta_2} > 0$, and $q(x) > 0$, then $\lambda_j > 0$.

Eigenfunctions:

For each eigenvalue λ_j, there exists an eigenfunction $\phi_j(x)$ (unique up to a multiplicative constant) such that:
(a) The eigenfunctions $\phi_j(x)$ are real and can be normalized to satisfy

$$\int_0^\ell r(x)\phi_j^2(x)dx = 1$$

(b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $r(x)$:

$$\langle \phi_j, \phi_k \rangle := \int_0^\ell r(x)\phi_j(x)\phi_k(x)dx = 0, \quad j \neq k$$

(c) Each eigenfunction $\phi_j(x)$ has exactly $j-1$ zeros in the interval $(0, \ell)$.

Expansion property: The eigenfunctions $\{\phi_j(x)\}$ form a complete set. Thus, if $f(x)$ is piecewise smooth, then we have

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$\text{with} \qquad c_n = \frac{\int_0^\ell r(x) f(x) \phi_n(x) dx}{\int_0^\ell r(x)\phi_n^2(x)dx}$$

Lagrange's Identity For sufficiently differentiable functions u and v, Lagrange's Identity states that

$$\int_0^\ell [v\mathcal{L}u - u\mathcal{L}v]dx = -\left. p(x)\left(u'v - uv'\right)\right|_0^\ell$$

Proof. Write

$$\begin{aligned}
\int_0^\ell v\mathcal{L}u dx &= \int_0^\ell v\left[-\left(pu'\right)' + qu\right] dx \\
&= -\left. vpu'\right|_0^\ell + \int_0^\ell pu'v'dx + \int_0^\ell quv dx \\
&= -\left. vpu'\right|_0^\ell + \left. upv'\right|_0^\ell + \int_0^\ell u\left[-\left(pv'\right)' + qv\right] dx \\
&= -\left. vpu'\right|_0^\ell + \left. upv'\right|_0^\ell + \int_0^\ell u\mathcal{L}v dx
\end{aligned}$$

Hence,

$$\int_0^\ell v\mathcal{L}u dx - \int_0^\ell u\mathcal{L}v dx = -\left. p(x)\left(u'v - uv'\right)\right|_0^\ell$$

If u and v satisfy the SL boundary conditions

$$\begin{aligned} \alpha_1 u(0) + \alpha_2 u'(0) = 0, \quad \beta_1 u(\ell) + \beta_2 u'(\ell) = 0 \\ \alpha_1 v(0) + \alpha_2 v'(0) = 0, \quad \beta_1 v(\ell) + \beta_2 v'(\ell) = 0 \end{aligned}$$

then the boundary terms cancel and

$$\int_0^\ell v\mathcal{L}u dx = \int_0^\ell u\mathcal{L}v dx$$

Example 8.1.1.

1. *We want to solve the eigenvalue problem*

$$(xy')' + \frac{2}{x}y = -\lambda\frac{1}{x}y, \quad x > 0$$

subject to the following boundary conditions

$$y'(1) = 0, \quad y'(2) = 0$$

Note that $r(x) = \frac{1}{x}$. *Expanding the derivative, we have*

$$xy'' + y' + \frac{2}{x}y = -\lambda\frac{1}{x}y$$

Multiply through by x *to obtain the Cauchy-Euler type equation:*

$$x^2y'' + xy' + (\lambda + 2)y = 0$$

The characteristic equation is

$$r^2 + \lambda + 2 = 0$$

Case 1: $\lambda + 2 < 0$**.**
We have two solutions $r_{1,2} = \pm\sqrt{-(\lambda+2)}$. *The general solution is then,*

$$y(x) = c_1 x^{-\sqrt{-(\lambda+2)}} + c_2 x^{\sqrt{-(\lambda+2)}}$$

The boundary conditions $y'(1) = y'(2) = 0$ *implies that* $c_1 = c_2 = 0$. *Trivial solution.*
Case 2: $\lambda + 2 = 0$**.**
We have a double solution $r_1 = r_2 = 0$. *The general solution is then,*

$$y(x) = c_1 + c_2 \ln x$$

The boundary conditions $y'(1) = 0$ *implies that* $c_2 = 0$ *and so* $y(x) = c_1$ *is a nontrivial solution.*
Case 3: $\lambda + 2 > 0$**.**
Thus, the general solution is

$$y(x) = c_1\cos(\sqrt{\lambda+2}\ln x) + c_2\sin(\sqrt{\lambda+2}\ln x)$$

Next we apply the boundary conditions. $y'(1) = 0$ *forces* $c_2 = 0$*. This leaves*

$$y(x) = c_1 \cos(\sqrt{\lambda+2}\ln x)$$

The second condition, $y'(2) = 0$*, yields*

$$\sin(\sqrt{\lambda+2}\ln 2) = 0$$

This will give nontrivial solutions when

$$\sqrt{\lambda+2}\ln 2 = n\pi, \quad n = 1, 2, 3 \ldots$$

In summary, the eigenfunctions for this eigenvalue problem are

$$y_n(x) = \cos\left(\frac{n\pi}{\ln 2}\ln x\right), \quad 1 \le x \le 2$$

and all (including $\lambda = -2$ *) the eigenvalues are* $\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2 - 2$ *for* $n = 0, 1, 2, \ldots$
We can check the orthogonality of eigenfunctions. We recall that

$$\langle y_n, y_m\rangle = \int_1^2 y_n(x) y_m(x) r(x) dx$$

Let $y = \pi\frac{\ln x}{\ln 2}$*. Then, we have:*

$$\begin{aligned}\langle y_n, y_m\rangle &= \int_1^2 \cos\left(\frac{n\pi}{\ln 2}\ln x\right)\cos\left(\frac{m\pi}{\ln 2}\ln x\right)\frac{dx}{x} \\ &= \frac{\ln 2}{\pi}\int_0^\pi \cos(ny)\cos(my)dy \\ &= \frac{\ln 2}{2}\delta_{n,m}\end{aligned}$$

where

$$\delta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

is the so called Kronecker delta.
Now let us consider expanding a function $f(x)$ *in terms of a "Fourier Series" of these new eigenfunctions in the following form*

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi \ln x}{\ln 2}\right)$$

In order to determine the coefficients c_n *we project the function* $f(x)$ *onto the basis functions* $\phi_n(x)$ *as follows:*

$$\begin{aligned}\int_1^2 f(x)\cos\left(\frac{m\pi\ln x}{\ln 2}\right)\frac{dx}{x} &= \sum_{n=0}^{\infty} c_n \int_1^2 \cos\left(\frac{m\pi\ln x}{\ln 2}\right)\cos\left(\frac{n\pi\ln x}{\ln 2}\right)\frac{dx}{x} \\ &= c_m\frac{\ln 2}{2}\end{aligned}$$

Hence,

$$c_n = \frac{2}{\ln 2}\int_1^2 f(x)\cos\left(\frac{n\pi \ln x}{\ln 2}\right)\frac{dx}{x}$$

Note

- If the operator $\mathcal{L}$ and the boundary conditions satisfy

$$\int_0^\ell v\mathcal{L}u dx = \int_0^\ell u\mathcal{L}v dx$$

then $\mathcal{L}$ is said to be self adjoint.

- With the notation $\langle f, g\rangle = \int_0^\ell f(x)g(x)dx$, the selfadjoint property can be written as

$$\langle v, \mathcal{L}u\rangle = \langle u, \mathcal{L}v\rangle$$

8.2 Application: Solving the heat equation with Robin boundary conditions

We consider the function $u(x,t)$, which models the temperature distribution in a heat-conductive rod of length L that is perfectly insulated along its sides. The left end of the rod is also perfectly insulated, meaning no heat escapes or enters at $x = 0$. Meanwhile, the right end at $x = L$ loses thermal energy at a rate proportional to its temperature at that point. The initial temperature distribution along the rod is given by the function $f(x)$.

Example 8.2.1. *The problem is stated as follows:*

$$\begin{aligned} u_t = \alpha^2 u_{xx}, \quad 0 < x < L,\ t > 0 \\ u_x(0,t) = 0, \quad u_x(L,t) + \kappa u(L,t) = 0 \\ u(x,0) = f(x) \end{aligned} \tag{8.4}$$

where $\kappa > 0$. We use the method of separation of variables to solve the problem (8.4). We first seek separated solutions of the form $u(x,t) = X(x)T(t)$ satisfying all of the homogeneous linear requirements, i.e. the first three conditions. Substituting the separated solution into the PDE yields

$$XT' = c^2X''T \Rightarrow \frac{X''}{X} = \frac{T'}{c^2T} = \lambda\ (constant)$$

since the two sides of the latter equation are functions of distinct independent variables. This gives us the pair of separated ODEs

$$X'' - \lambda X = 0, \quad T' - \lambda c^2 T = 0$$

From the first boundary condition we obtain

$$X'(0)T(t) = 0 \Rightarrow X'(0) = 0$$

since we do not want $T \equiv 0$. The second boundary condition requires that

$$X'(L)T(t) = -\kappa X(L)T(t) \Rightarrow X'(L) = -\kappa X(L)$$

We have thus obtained an ODE boundary value problem in X that requires a case by case analysis of the possible values of k for which there are nontrivial (nonzero) solutions.

Case 1: $\lambda = \mu^2 > 0$.

In this situation the ODE for X becomes

$$X'' - \mu^2 X = 0$$

with characteristic equation

$$r^2 - \mu^2 = 0$$

whose roots are $r = \pm\mu$. The solutions are then given by

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}$$

The boundary conditions require that

$$\begin{aligned}
&\mu\left(c_1 e^{\mu\cdot 0} - c_2 e^{-\mu\cdot 0}\right) = 0 \Rightarrow c_1 - c_2 = 0\\
&\mu\left(c_1 e^{\mu L} - c_2 e^{-\mu L}\right) = -\kappa\left(c_1 e^{\mu L} + c_2 e^{-\mu L}\right) \Rightarrow c_1(\kappa+\mu)e^{\mu L} + c_2(\kappa-\mu)e^{-\mu L} = 0
\end{aligned}$$

or in matrix form

$$\begin{pmatrix} 1 & -1 \\ (\kappa+\mu)e^{\mu L} & (\kappa-\mu)e^{-\mu L} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is

$$(\kappa-\mu)e^{-\mu L} + (\kappa+\mu)e^{\mu L} = \kappa\left(e^{\mu L} + e^{-\mu L}\right) + \mu\left(e^{\mu L} - e^{-\mu L}\right) > 0$$

which means that the only possibility is that $c_1 = c_2 = 0$, or in other words $X \equiv 0$.

Case 2: $\lambda = 0$.

Now the ODE in X simplifies to $X'' = 0$ which means that $X = ax + b$. The first boundary condition immediately implies $a = 0$ and the second then becomes

$$0 = -\kappa b \Rightarrow b = 0$$

which once again tells us that $X \equiv 0$. So we move on.

Case 3: $\lambda = -\mu^2 < 0$.

Things finally get interesting. The ODE becomes $X'' + \mu^2 X = 0$ whose characteristic equation has roots $\pm i\mu$, so that $X(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$. From the first boundary condition we find

$$-c_1\sin 0 + c_2\cos 0 = 0 \Rightarrow c_2 = 0$$

The second boundary condition is then

$$-\mu c_1 \sin(\mu L) = -\kappa c_1 \cos(\mu L) \Rightarrow \tan\mu L = \frac{\kappa}{\mu}$$

since we do not want $c_1 = 0$ at this point. This equation has an increasing sequence of positive solutions (We have proved it using a graphic), which we label

$$0 < \mu_1 < \mu_2 < \mu_3 < \cdots$$

Finally, we obtain the nontrivial solutions

$$X_n = \cos(\mu_n x), \quad n \in \mathbb{N}$$

Since $-\mu^2 = \lambda$*, for each* $n \in N$ *the ODE for* T *becomes*

$$T' + (c\mu_n)^2 T = 0 \Rightarrow T' = -(c\mu_n)^2 T \Rightarrow T = T_n = c_n e^{-(c\mu_n)^2 t}$$

We have finally obtained our separated solutions:

$$u_n(x,t) = c_n e^{-(c\mu_n)^2 t} \cos(\mu_n x), \quad n \in \mathbb{N}$$

where μ_n *is the* n*th positive solution to* $\tan \mu L = \kappa/\mu$.
Hence,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-(c\mu_n)^2 t} \cos(\mu_n x)$$

Using the initial condition, we have

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-(c\mu_n)^2 \cdot 0} \cos(\mu_n x) = \sum_{n=1}^{\infty} c_n \cos(\mu_n x) \tag{8.5}$$

Now, we want to show that the functions $\cos \mu_n x$ *are pairwise orthogonal on the interval* $[0, L]$*. Assume that for different eigenvalues* μ_n^2 *and* μ_m^2 *the corresponding eigenfunctions are given by*

$$X_n(x) = \cos(\mu_n x) \quad \textit{and} \quad X_m(x) = \cos(\mu_m x)$$

with $\mu_n \neq \mu_m$.
Our goal is to show that these eigenfunctions are orthogonal:

$$\int_0^L X_n(x) X_m(x) dx = 0 \quad \textit{for } n \neq m$$

We know that $X_n(x)$ *satisfies*

$$X_n''(x) + \mu_n^2 X_n(x) = 0$$

If we multiply the latter identity by $X_m(x)$ *and integrate over* $[0, L]$ *we obtain:*

$$\int_0^L X_m(x) X_n''(x) dx + \mu_n^2 \int_0^L X_m(x) X_n(x) dx = 0$$

Let the first term in the above equation be:

$$I = \int_0^L X_m(x) X_n''(x) dx$$

An integration by parts gives:

$$I = [X_m(x) X_n'(x)]_0^L - \int_0^L X_m'(x) X_n'(x) dx$$

Using the boundary conditions, we get

$$I = -\kappa X_m(L) X_n(L) - \int_0^L X_m'(x) X_n'(x) dx$$

Returning to Substituting back we now have:

$$-\kappa X_m(L) X_n(L) - \int_0^L X_m'(x) X_n'(x) dx + \mu_n^2 \int_0^L X_m(x) X_n(x) dx = 0$$

A second integration by parts along with the equation solved by X_m *lead us to*

$$\left(\mu_n^2 - \mu_m^2\right) \int_0^L X_n(x) X_m(x) dx = 0$$

Since $\mu_n^2 \neq \mu_m^2$ *for* $n \neq m$*, we must have*

$$\int_0^L X_n(x) X_m(x) dx = 0$$

Finally, we multiply equation (8.5) by $\cos(\mu_m x)$ *and integrate over* $[0, L]$*. Thanks to the orthogonality relation, we obtain*

$$c_n = \frac{\int_0^L f(x) \cos(\mu_n x) dx}{\int_0^L \cos^2(\mu_n x) dx}$$

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