1. Consider the differential equation

$$Ly = \quad 2x^2y'' - xy' + (1+x^2)y = 0 \qquad (1)$$

(a) Classify the points $0 \le x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

a) $X=0$ IS A SINGULAR POINT AND ALL $X: 0<X<\infty$ ARE ORDINARY POINTS

FOR $X=0$: $\lim_{X\to 0} X\left(\frac{-X}{2X^2}\right) = -\frac{1}{2} = p_0$ AND $\lim_{X\to 0} X^2\frac{(1+X^2)}{2X^2} = \frac{1}{2} = q_0$. SINCE $|p_0|,|q_0|<\infty$ $X=0$ IS A REGULAR SINGULAR POINT.

b) CONSIDER THE CAUCHY EULER EQ:

$$x^2y'' - \frac{x}{2}y' + \frac{y}{2} = 0$$

LET $y = x^r \Rightarrow 2r(r-1) - r + 1 = 2r^2 - 3r + 1 = (2r-1)(r-1) = 0 \Rightarrow r = \frac{1}{2}, 1.$

c) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$ $\quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$0 = Ly = 2x^2y'' \qquad -xy' \qquad +y \qquad +x^2y$$

$$= \underbrace{\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}_{m=n} + \sum_{n=0}^{\infty} a_n x^{n+r+2}.$$

$m=n+2 \quad n=m-2$

$n=0 \Rightarrow m=2$

$$= \sum_{m=0}^{\infty} a_m\left[(m+r)\{2(m+r-1)-1\}+1\right]x^{m+r} + \sum_{m=2}^{\infty} a_{m-2}x^{m+r}$$

$$0 = a_0[2r(r-1)-r+1]x^r + a_1[2(r+1)r-(r+1)+1]x^{r+1} + \sum_{m=2}^{\infty}\left[a_m\left[(m+r)\{2(m+r-1)-1\}+1\right]+a_{m-2}\right]x^{m+r}$$

$x^r]$ $\quad a_0[2r^2-3r+1] = a_0(2r-1)(r-1) = 0 \quad r=\frac{1}{2}, r=1$ AS ABOVE

$x^{r+1}]$ $\quad a_1[2r^2+r] = 0 \quad$ IF $r=\frac{1}{2}$ $2r^2+r=1$ $\quad r=1$ $2r^2+r=3$ $\quad$ IN BOTH CASES WE MUST HAVE $a_1=0$

$x^{m+r}{}_{m\geq 2}]$ RECURSION $\quad a_m = -a_{m-2}/\left[(m+r)\{2(m+r-1)-1\}+1\right] \quad m\geq 2. \quad (*)$

$r=\frac{1}{2}$: $\quad a_m = -a_{m-2}/\left[(m+\frac{1}{2})(2m+\cancel{1}-2-\cancel{1})+1\right] = -a_{m-2}/\left[(2m+1)(m-1)+1\right] = \dfrac{-a_{m-2}}{m(2m-1)}$

$a_2 = -a_0/2.3 \qquad a_4 = -a_2/4.7 = +a_0/168$

$\therefore \quad y_0(x) = a_0 x^{1/2}[1 - x^2/6 + x^4/168 - \cdots]$

$r=1$: $\quad a_m = -a_{m-2}/\left[(m+1)(2m-1)+1\right] = -a_{m-2}/m(2m+1)$

$a_2 = -a_0/2.5 \qquad a_4 = -a_2/4.9 = +a_0/360$

$\therefore \quad y_1(x) = a_0 x[1 - x^2/10 + x^4/360 - \cdots]$

NOTE SINCE $a_1 = 0$ THE RECURSION (*) IMPLIES $0 = a_3 = a_5 = \cdots$

2. Consider a conducting metal bar of length $\pi/2$ whose initial temperature is $u(x,0)=x$ and which loses heat to its surroundings. Assume that the left end of the bar is maintained at a zero temperature while the right end is insulated. The temperature distribution in the bar $u(x,t)$ is determined by the following initial boundary value problem for the heat equation:

$$\begin{aligned} u_t &= u_{xx}-u\,, \qquad 0<x<\pi/2\,, \qquad t>0 \\ u(0,t) &= 0, \qquad u_x(\pi/2,t)=0 \\ u(x,0) &= x \end{aligned} \tag{2}$$

(a) Determine the solution to the boundary value problem (2) by separation of variables.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2,\Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n,t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(\pi/2,t)=0$ with $O(\Delta x^2)$ accuracy.

Hint: Taylor's expansion may prove useful: $f(x+\Delta x)=f(x)+\frac{f'(x)}{1!}\Delta x+\frac{f''(x)}{2!}\Delta x^2+O(\Delta x^3)$.

[6 marks]

[total 20 marks]

a) LET $u(x,t)=X(x)T(t)$ AND SUBSTITUTE INTO THE PDE:

$X\dot{T} = X''T - XT$ WHERE $\dot{T}(t)=\frac{dT}{dt}$ & $X'(x)=\frac{dX}{dx}$

$\boxed{\div XT}$ $\frac{\dot{T}}{T}+1 = \frac{X''}{X} = -\lambda^2$ A CONSTANT.

$T]$ $\dot{T} = -(1+\lambda^2)T \Rightarrow T(t)=Ce^{-(1+\lambda^2)t}$.

$X]$ $X''+\lambda^2 X = 0 \qquad X(0)=0=X'(\pi/2)$

$X = A\cos\lambda x + B\sin\lambda x \qquad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$0 = X(0) = A.1 \Rightarrow A=0;\ 0 = X'(\pi/2) = B\lambda\cos(\lambda\pi/2) \Rightarrow \lambda_n = (2n+1)\ \ n=0,1,2.$

$X_n = \sin(2n+1)x$

$\therefore\ u(x,t) = \sum_{n=0}^{\infty} b_n e^{-(1+\lambda_n^2)t}\sin\lambda_n x$

$x = u(x,0) = \sum_{n=0}^{\infty} b_n \sin(2n+1)x$

$\therefore\ b_n = \frac{2}{\pi/2}\int_0^{\pi/2} x\sin(2n+1)x\,dx = \frac{4}{\pi}\Big[-\frac{x\cos(2n+1)x}{(2n+1)}\Big|_0^{\pi/2} + \frac{1}{(2n+1)}\int_0^{\pi/2}\cos(2n+1)x\,dx\Big]$

$= \frac{4}{\pi}\Big\{\frac{\sin(2n+1)x}{(2n+1)^2}\Big|_0^{\pi/2}\Big\} = \frac{4}{\pi}\frac{(-1)^n}{(2n+1)^2}$

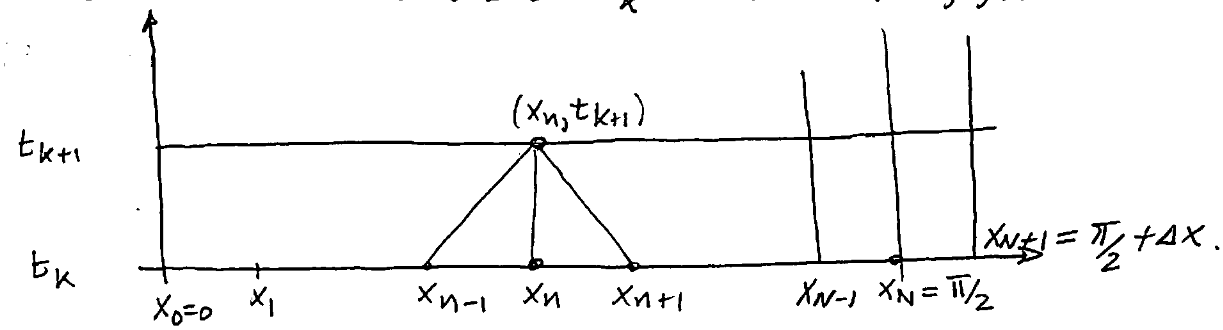
$\therefore\ u(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}e^{-[1+(2n+1)^2]t}\sin(2n+1)x$

(Question 2 Continued)

DIVIDE THE SPATIAL DOMAIN $[0, \pi/2]$ INTO N+1 MESH POINTS

$$x_n = n\Delta x \qquad n = 0, 1, \ldots, N \qquad \Delta x = \pi/2N.$$

AND DEFINE THE TIME STEPS $t_k = \Delta t . k \quad k = 0, 1, \ldots$



USING TAYLOR'S THM

$$u(x \pm \Delta x, t) = u(x,t) \pm \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{\Delta x^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{24}\frac{\partial^4 u}{\partial x^4} \pm \cdots \qquad (1\pm)$$

$$\frac{(1+)+(1-)}{\Delta x^2} \Rightarrow \quad \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2} + O(\Delta x^2)$$

$$u(x, t+\Delta t) = u(x,t) + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + \cdots$$

$$\therefore \quad \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \frac{\partial u}{\partial t} + O(\Delta t)$$

$$\therefore \quad u_t \simeq \frac{u_n^{k+1} - u_n^k}{\Delta t} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} - u_n^k$$

$$\boxed{\therefore u_n^{k+1} = u_n^k + \frac{\Delta t}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) - \Delta t\, u_n^k.}$$

TO APPROXIMATE THE DERIVATIVE AT $x_N = \pi/2$ USE $\frac{(1+)-(1-)}{2\Delta x}$

$$\frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\Delta x} = \frac{\partial u}{\partial x} + O(\Delta x^2)$$

CHOOSE $x = \pi/2 = x_N$ AND INTRODUCING THE GHOST MESH POINT $x_{N+1} = \pi/2 + \Delta x$

WE HAVE $$\frac{u_{N+1} - u_{N-1}}{2\Delta x} = 0$$

OR $\boxed{u_{N+1} = u_{N-1}}$

3. The motion of a damped string subject to an imposed load satisfies the following initial-boundary value problem:

$$u_{tt} + 2\gamma u_t = u_{xx} - 8\sin x\cos x,\ 0 < x < \pi,\ t > 0 \quad (3)$$
$$u(0,t) = u(\pi,t) = 0 \quad (4)$$
$$u(x,0) = 0,\ u_t(x,0) = \sin 3x.$$

(a) Determine the static deflection $w(x)$ of the string (i.e., the steady solution), which is determined by solving (3) with $u_{tt} = u_t = 0$ and subject to the boundary conditions (4).

[5 marks]

(b) Let $u(x,t) = w(x) + v(x,t)$ and determine the corresponding boundary value problem for $v(x,t)$.

[5 marks]

(c) Assuming that $\gamma < 1$ use the method of separation of variables to solve for $v(x,t)$ and therefore $u(x,t)$.

[6 marks]

(d) Now assuming no damping, i.e., letting $\gamma = 0$, use D'Alembert's solution (see the formula sheet) to determine $v(x,t)$ and therefore $u(x,t)$.

[4 marks]
[total 20 marks]

a) LET W(x) BE THE STEADY SOL'N

$0 = W_{xx} - 8\sin x\cos x \Rightarrow W_{xx} = 4\sin 2x \Rightarrow W_x = -2\cos 2x \quad \boxed{W(x) = -\sin 2x}$

NOTE $W(0)=0$ AND $W(\pi)=0$.

b) $\cancel{W_{tt}}^{0} + V_{tt} + 2\gamma(\cancel{W_t}^{0} + V_t) = \{W_{xx} - 8\sin x\cos x\} + V_{xx} \Rightarrow \boxed{V_{tt} + 2\gamma V_t = V_{xx}}$

BC. $0 = u(0,t) = \cancel{W(0)} + V(0,t) \Rightarrow \boxed{V(0,t)=0} \quad 0 = u(\pi,t) = \cancel{W(\pi)} + V(\pi,t) \Rightarrow \boxed{V(\pi,t)=0}$

IC: $0 = u(x,0) = W(x) + V(x,0) = -\sin 2x + V(x,0) \Rightarrow \boxed{V(x,0) = \sin 2x}$

$\sin 3x = u_t(x,0) = \cancel{W_t}^{0} + V_t(x,0) \Rightarrow \boxed{V_t(x,0) = \sin 3x}$

c) LET $V(x,t) = X(x)T(t) \Rightarrow X\ddot{T} + 2\gamma X\dot{T} = X''T \Rightarrow \dfrac{\ddot{T}+2\gamma\dot{T}}{T(t)} = \dfrac{X''}{X} = -\lambda^2$ CONST

X] $X'' + \lambda^2 X = 0,\ X(0) = 0 = X(\pi) \Rightarrow \lambda = \frac{n\pi}{\pi} = n,\ n = 1,2\ldots\ X_n = \sin(nx)$

T] $\ddot{T} + 2\gamma\dot{T} - n^2T = 0$ LET $T(t) = e^{rt} \Rightarrow r^2 + 2\gamma r - n^2 = 0$

OR $r = \{-2\gamma \pm \sqrt{4\gamma^2 - 4n^2}\}/2 = -\gamma \pm \sqrt{\gamma^2 - n^2} = -\gamma \pm i\sqrt{n^2-\gamma^2} = -\gamma \pm i\mu_n$ SINCE $\gamma<1$

$\mu_n = \sqrt{n^2-\gamma^2}$

$\therefore T_n(t) = e^{-\gamma t}[A_n\cos\mu_n t + B_n\sin\mu_n t]$

$\therefore V(x,t) = \sum_{n=1}^{\infty} e^{-\gamma t}[A_n\cos\mu_n t + B_n\sin\mu_n t]\sin(nx)$

$V_t(x,t) = \sum_{n=1}^{\infty} e^{-\gamma t}\{(-\gamma)[A_n\cos\mu_n t + B_n\sin\mu_n t] + [A_n(-\mu_n)\sin\mu_n t + B_n\mu_n\cos\mu_n t]\}\sin(nx)$

$\sin 2x = V(x,0) = \sum_{n=1}^{\infty} A_n\sin nx \Rightarrow A_n = \begin{cases} 0 & n\neq 2 \\ 1 & n=2 \end{cases} = \delta_{n2}$

$\sin 3x = \sum_{n=1}^{\infty}[-\gamma[\delta_{n2}] + B_n\mu_n]\sin nx \Rightarrow B_2 = \gamma/\mu_2 \quad B_3 = \frac{1}{\mu_3} \quad B_n = 0\ n\neq 2,3$

$\therefore u(x,t) = -\sin 2x + e^{-\gamma t}\{[\cos\mu_2 t + \frac{\gamma}{\mu_2}\sin\mu_2 t]\sin(2x) + \frac{1}{\mu_3}\sin\mu_3 t\sin 3x\}$

4. Consider the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0 \qquad (1)$$
$$y'(1) = 0 = y(e^{\pi/2}) \qquad (2)$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < e^{\pi/2}, \quad 0 < \theta < \pi$$
$$u(r,0) = 0 \text{ and } u(r,\pi) = f(r)$$
$$\frac{\partial u(1,\theta)}{\partial r} = 0 \text{ and } u(e^{\pi/2},\theta) = 0$$

[12 marks]
[total 20 marks]

a) $F(x) = \frac{1}{x^2}e^{\int \frac{x}{x^2}dx} = \frac{1}{x^2}e^{\ln x} = 1/x$.

$(1)\times\frac{1}{x} \Rightarrow \quad -xy'' + y' = \boxed{-(xy')' = \lambda y/x.}$ WHICH IS IN S-L FORM.

NOW $x^2y'' + xy' + \mu^2 y = 0$ (WHERE $\lambda = \mu^2$) IS A CAUCHY EULER EQ SO LET

$y = x^\gamma \Rightarrow \gamma(\gamma-1) + \gamma + \mu^2 = 0 \quad \gamma = \pm i\mu.$

$\therefore \quad y(x) = c_1 x^{i\mu} + c_2 x^{-i\mu} = A\cos\mu\ln x + B\sin\mu\ln x$ SINCE $x^{i\mu} = e^{i\mu\ln x}$

$y'(x) = -A\sin\mu\ln x.(\mu/x) + B\cos\mu\ln x.(\mu/x)$

$0 = y'(1) = -A.0 + B.1.\mu/1 \Rightarrow B = 0$

$0 = y(e^{\pi/2}) = A\cos\mu\ln e^{\pi/2} = A\cos(\mu\pi/2) \Rightarrow \mu_n = (2n+1) \quad n = 1,2\ldots$

$y_n(x) = \cos(2n+1)\ln x.$

b) LET $u(r,\theta) = R(r)\Theta(\theta) \Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$

$\times\frac{r^2}{R\Theta}\Big] \quad \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = -\mu^2$ (− SIGN BECAUSE HOMOG. BC IN R)

$\Theta\big] \ \Theta'' - \mu^2\Theta = 0 \quad \Theta(0) = 0 \Rightarrow \Theta = A\cosh\mu\theta + B\sinh\mu\theta$

$0 = \Theta(0) = A \Rightarrow A = 0 \qquad \Theta(\theta) = B\sinh\mu\theta.$

$R\big] \ r^2R'' + rR' - \mu^2R = 0 \qquad R'(1) = 0 = R(e^{\pi/2}) \Rightarrow \mu_n = (2n+1) \quad R_n(r) = \cos(2n+1)\ln r$
$n = 0,1,\ldots$

$$\therefore \boxed{u(r,\theta) = \sum_{n=0}^{\infty} B_n \sinh\mu_n\theta\cos(\mu_n\ln r)}$$

$$f(r) = u(r,\pi) = \sum_{n=0}^{\infty} B_n\sinh(\mu_n\pi)\cos(\mu_n\ln r) = \sum_{n=0}^{\infty} b_n\cos(\mu_n\ln r)$$

$$\therefore \int_1^{e^{\pi/2}} \frac{f(r)}{r}\cos(\mu_m\ln r)dr = \sum_{n=0}^{\infty} b_n\int_1^{e^{\pi/2}}\cos(\mu_m\ln r)\cos(\mu_n\ln r)\frac{dr}{r}$$

$\boxed{x = \ln r \quad dx = dr/r \quad r=1 \Rightarrow x=0,\ r=e^{\pi/2} \Rightarrow x = \pi/2}$

$$\text{NOW} \int_1^{e^{\pi/2}}\cos(\mu_m\ln r)\cos(\mu_n\ln r)\frac{dr}{r} = \int_0^{\pi/2}\cos(\mu_m x)\cos(\mu_n x)dx = \delta_{mn}\frac{(\pi/2)}{2}$$

$$\therefore \boxed{B_n = \frac{b_n}{\sinh(\mu_n\pi)} = \frac{4}{\pi\sinh(\mu_n\pi)}\int_1^{e^{\pi/2}}\frac{f(r)}{r}\cos(\mu_n\ln r)dr}$$

5. **Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:**

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + 1 - xe^{-t},\ 0<x<1,\ t>0 \\ u_x(0,t) &= e^{-t},\ \text{and } u(1,t) = t \\ u(x,0) &= x. \end{aligned}$$

[20 marks]

FIRSTLY LOOK FOR A FUNCTION $W(x,t) = A(t)x + B(t)$ THAT SATISFIES THE NON-ZERO BC

$e^{-t} = W_x = A(t)$; $t = W(1) = e^{-t}\cdot 1 + B(t) \Rightarrow B(t) = t - e^{-t}$

$\therefore$ $\boxed{W(x,t) = e^{-t}x + t - e^{-t}}$ SATISFIES THE BC.

NOW LET $u(x,t) = W(x,t) + V(x,t)$ AND DETERMINE THE BOUNDARY VALUE PROBLEM FOR V.

PDE: $u_t = W_t + V_t = -\cancel{e^{-t}x} + \cancel{1} + e^{-t} + V_t = \alpha^2(\cancel{W_{xx}} + V_{xx}) + \cancel{1} - \cancel{xe^{-t}} \Rightarrow V_t = \alpha^2 V_{xx} - e^{-t}$

BC: $\cancel{e^{-t}} = u_x(0,t) = W_x(0,t) + V_x(0,t) = \cancel{e^{-t}} + V_x(0,t) \Rightarrow V_x(0,t) = 0$

$\cancel{t} = u(1,t) = W(1,t) + V(1,t) = \cancel{t} + V(1,t) \Rightarrow V(1,t) = 0$

IC: $\cancel{x} = u(x,0) = W(x,0) + V(x,0) = \cancel{x} - 1 + V(x,0) \Rightarrow V(x,0) = 1$

SINCE V IS SUBJECT TO HOMOGENEOUS BC WE CAN USE AN EIGENFUNCTION EXPANSION

THE EIGENVALUES ASSOCIATED WITH THE EIGENVALUE PROBLEM $X'' + \lambda^2 X = 0$ $X'(0) = 0 = X(1)$

ARE $\lambda_n = (2n+1)\pi/2$ $n = 0, 1, 2 ..$ AND THE CORRESPONDING EIGENFUNCTIONS ARE $X_n = \cos\lambda_n x$.

EXPAND THE SINK TERM $S(x,t) = -e^{-t} = \sum_{n=0}^{\infty} \hat{S}_n(t)\cos\lambda_n x$

$\therefore$ $\hat{S}_n(t) = \frac{2}{1}\int_0^1 (-e^{-t})\cos\lambda_n x\,dx = -2e^{-t}\frac{\sin(\lambda_n x)}{\lambda_n}\Big|_0^1 = \frac{-2}{\lambda_n}\sin\left(\frac{(2n+1)\pi}{2}\right)e^{-t} = \frac{(-1)^{n+1}\cdot 2}{\lambda_n}e^{-t} = C_n e^{-t}$

LET $V(x,t) = \sum_{n=0}^{\infty}\hat{V}_n(t)\cos\lambda_n x$ $\quad \frac{\partial V}{\partial t} = \sum_{n=0}^{\infty}\frac{d\hat{V}_n}{dt}\cos\lambda_n x$ $\quad \frac{\partial^2 V}{\partial x^2} = \sum_{n=0}^{\infty}\hat{V}_n(-\lambda_n^2)\cos\lambda_n x$

$\therefore$ $0 = V_t - \alpha^2 V_{xx} + e^{-t} = \sum_{n=0}^{\infty}\left\{\frac{d\hat{V}_n}{dt} + \alpha^2\lambda_n^2\hat{V}_n - C_n e^{-t}\right\}\cos(\lambda_n x)$ SINCE $\cos\lambda_n x$ ARE LINEARLY INDEP. IT FOLLOWS THAT $\{\ \} = 0$

$\therefore$ $\frac{d\hat{V}_n}{dt} + \alpha^2\lambda_n^2\hat{V}_n = C_n e^{-t} \Rightarrow \frac{d}{dt}\left[e^{+\alpha^2\lambda_n^2 t}\hat{V}_n\right] = C_n e^{(\alpha^2\lambda_n^2 - 1)t}$

INTEGRATE $e^{+\alpha^2\lambda_n^2 t}\hat{V}_n = \frac{C_n}{\alpha^2\lambda_n^2 - 1}e^{(\alpha^2\lambda_n^2 - 1)t} + d_n$

$\therefore$ $\hat{V}_n(t) = \frac{C_n}{\alpha^2\lambda_n^2 - 1}e^{-t} + d_n e^{-\alpha^2\lambda_n^2 t}$

$\therefore$ $V(x,t) = \sum_{n=0}^{\infty}\left[\frac{C_n e^{-t}}{\alpha^2\lambda_n^2 - 1} + d_n e^{-\alpha^2\lambda_n^2 t}\right]\cos\lambda_n x$

NOW $1 = V(x,0) = \sum_{n=0}^{\infty}\left[\frac{C_n\cdot 1}{\alpha^2\lambda_n^2 - 1} + d_n\right]\cos\lambda_n x = \sum_{n=0}^{\infty} a_n\cos\lambda_n x$

WHERE $a_n = \frac{2}{1}\int_0^1 1\cos\lambda_n x\,dx = 2\frac{\sin\lambda_n x}{\lambda_n}\Big|_0^1 = \frac{2\sin\left(\frac{(2n+1)\pi}{2}\right)}{\lambda_n} = -C_n$

$\therefore$ $-C_n = \frac{C_n}{\alpha^2\lambda_n^2 - 1} + d_n \Rightarrow d_n = -C_n\left[\frac{\alpha^2\lambda_n^2 - 1 + 1}{\alpha^2\lambda_n^2 - 1}\right] = -\frac{\alpha^2\lambda_n^2 C_n}{\alpha^2\lambda_n^2 - 1}$

$\therefore$ $u(x,t) = e^{-t}(x-1) + t + \sum_{n=0}^{\infty}\frac{C_n}{\alpha^2\lambda_n^2 - 1}\left[e^{-t} - \alpha^2\lambda_n^2 e^{-\alpha^2\lambda_n^2 t}\right]\cos(\lambda_n x)$

WHERE $C_n = \frac{(-1)^{n+1}2}{\lambda_n}$ AND $\lambda_n = \frac{(2n+1)\pi}{2}$ $n = 0, 1, ..$