1. Consider the differential equation

$$Ly = 2x^2(1-x)y'' - 3xy' + 2y = 0 \qquad (1)$$

(a) Classify the points $-\infty < x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) What form of expansion would you use around the point $x_0 = 1$? What is the minimal radius of convergence of this series?

(c) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(d) Use the series expansion in (c) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[total 20 marks]

a) $X \in \mathbb{R}\setminus\{0,1\}$ ARE ORDINARY POINTS $X=0$ & 1 ARE SINGULAR POINTS.

$X=1$: $\lim_{X\to 1}(X-1)\cdot\frac{-3X}{2X^2(1-X)} = \frac{+3}{2} = p_0 < \infty$ $\lim_{X\to 1}(X-1)^2\frac{2}{2X^2(1-X)} = 0 < \infty \Rightarrow X=1$ IS A REGULAR SINGULAR PT

$X=0$: $\lim_{X\to 0} X\cdot\frac{-3X}{2X^2(1-X)} = -\frac{3}{2} = p_0$ $\lim_{X\to 0} X^2\frac{2}{2X^2(1-X)} = 1 = q_0$ $|q_0|<\infty$ & $|p_0|<\infty \Rightarrow X=0$ IS A R.S.P.

b) SINCE $X=1$ IS A RSP ASSUME $y(x) = \sum_{n=1}^{\infty} a_n(x-1)^{n+r}$. SINCE THE CLOSEST SINGULAR POINT IS AT $X=0$ THE RADIUS OF CONVERGENCE $\rho \geq 1 = |1-0|$

c) THE INDICIAL EQ (FROM a) IS $r(r-1)-\frac{3}{2}r+1=0 \Rightarrow 2r^2-5r+2=(2r-1)(r-2)=0$ $r=\frac{1}{2}, 2$.

d) $y(x)=\sum_{n=0}^{\infty} a_n x^{n+r}$ $y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$ $y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$

$$Ly = 2x^2y'' \quad -2x^3y'' \quad -3xy' \quad +2y = 0$$

$$= \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r+1} - \sum_{n=0}^{\infty} 3a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$m=n$ $m=n+1$, $n=m-1$, $n=0 \Rightarrow m=1$ $m=n$ $m=n$

$$= \sum_{m=0}^{\infty} a_m[(m+r)\{2(m+r-1)-3\}+2]x^{m+r} - \sum_{m=1}^{\infty} 2a_{m-1}(m+r-1)(m+r-2)x^{m+r} = 0$$

$$= a_0\{r[2(r-1)-3]+2\}x^r + \sum_{m=1}^{\infty}\left(a_m[(m+r)\{2(m+r)-5\}+2] - 2a_{m-1}(m+r-1)(m+r-2)\right)x^{m+r} = 0$$

$x^r]$ $2r^2-5r+2=(2r-1)(r-2)=0$ $r=\frac{1}{2}, 2$.

x^{m+r} $m\geq 1]$ $a_m = \frac{2a_{m-1}(m+r-1)(m+r-2)}{(m+r)\{2(m+r)-5\}+2}$

$r=\frac{1}{2}$: $a_m = \frac{2a_{m-1}(m-\frac{1}{2})(m-\frac{3}{2})}{(m+\frac{1}{2})2(m-2)+2} = \frac{\frac{1}{2}a_{m-1}(2m-1)(2m-3)}{(2m+1)(m-2)+2} = \frac{\frac{1}{2}a_{m-1}(2m-1)(2m-3)}{2m^2-3m-\cancel{2}+\cancel{2}} = \frac{a_{m-1}(2m-1)\cancel{(2m-3)}}{2m\cancel{(2m-3)}}$

$a_1 = \frac{a_0}{2}$ $a_2 = \frac{a_1\cdot 3}{4} = \frac{3a_0}{8}$

$y_1(x) = x^{1/2}[1 + \frac{x}{2} + \frac{3x^2}{8} + \dots]$

$r=2$: $a_m = \frac{2a_{m-1}(m+1)m}{(m+2)\{2m-1\}+2} = \frac{2a_{m-1}(m+1)m}{2m^2+3m-\cancel{2}+\cancel{2}} = \frac{2a_{m-1}(m+1)}{(2m+3)}$

$a_1 = \frac{2a_0\cdot 2}{5} = \frac{4a_0}{5}$ $a_2 = \frac{2\cdot 3a_1}{7} = \frac{24}{35}a_0$

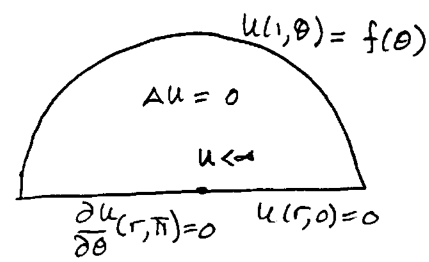
$y_2(x) = x^2[1 + \frac{4x}{5} + \frac{24}{35}x^2 + \dots]$

2. Solve the following mixed boundary value problem for Laplace's equation on the semi-circular region:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < 1, \quad 0 < \theta < \pi \\ u(r,0) &= 0 \quad \text{and} \quad \frac{\partial u(r,\pi)}{\partial \theta} = 0 \\ u(1,\theta) &= f(\theta) \quad \text{and} \quad u(r,\theta) < \infty \quad \text{as } r \to 0 \end{aligned}$$

[total 20 marks]

LET $u(r,\theta) = R(r)\Theta(\theta)$



$$\Delta u = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$\times \frac{r^2}{R\Theta}$ AND REARRANGE: $\frac{r^2R'' + rR'}{R(r)} = -\frac{\Theta''}{\Theta} = \mu^2$

$\Theta]$ $\left.\begin{matrix} \Theta'' + \mu^2\Theta = 0 \\ \Theta(0) = 0 = \Theta'(\pi) \end{matrix}\right\}$ $\mu_n = \frac{(2n-1)\pi}{2\pi} = \frac{(2n-1)}{2}$ $n = 1, 2, \ldots$ $\Theta_n = \sin\frac{(2n-1)}{2}\theta$

$R]$ $r^2R'' + rR' - \mu^2 R = 0$ THIS IS A CAUCHY EULER EQ SO LET $R(r) = r^\gamma$

$\lim_{r\to 0} R(r) < \infty$. $\Rightarrow \gamma(\gamma - 1) + \gamma - \mu^2 = \gamma^2 - \mu^2 = 0$ $\gamma = \pm\mu$

$\therefore R(r) = Ar^{\mu} + Br^{-\mu}$

SINCE $R(r) < \infty$ AS $r \to 0$ IT FOLLOWS THAT $B = 0$

$$\therefore u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\mu_n} \sin\mu_n\theta$$

FINAL BC:

$$f(\theta) = u(1,\theta) = \sum_{n=1}^{\infty} A_n 1^{\mu_n} \sin\mu_n\theta$$

THUS $A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin\frac{(2n-1)}{2}\theta\, d\theta$

$$\therefore u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{(2n-1)}{2}} \sin\frac{(2n-1)}{2}\theta$$

3. Solve the telegraph equation with $0 < \gamma < 1$ subject to an exponentially decaying forcing function:

$$u_{tt} + 2\gamma u_t = u_{xx} + e^{-2t}\cos(5x), \ 0 < x < \pi/2, \ t > 0$$
$$u_x(0,t) = 0 \text{ and } u(\pi/2, t) = 0, \quad t > 0$$
$$u(x,0) = 0, \qquad u_t(x,0) = \cos(3x), \quad 0 < x < \pi/2$$

THE EIGENVALUE PROBLEM ASSOCIATED WITH THESE HOMOGENEOUS BC [total 20 marks]

IS $\left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X'(0) = 0 = X(\pi/2) \end{array}\right\} \Rightarrow \mu_n = \dfrac{(2n-1)\pi}{2(\pi/2)} = (2n-1) \quad n = 1, 2, \ldots$ ARE THE EIGENVALUES

$X_n = \cos(2n-1)x$ ARE THE EIGENFUNCTIONS

LET $S(x,t) = e^{-2t}\cos 5x = \sum_{n=1}^{\infty} S_n(t)\cos(2n-1)x \Rightarrow S_n(t) = e^{-2t}\delta_{n3}$

NOW ASSUME $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\cos\mu_n x \qquad u_{xx} = \sum_{n=1}^{\infty} u_n\{-\mu_n^2\}\cos\mu_n x$

$0 = u_{tt} + 2\gamma u_t - u_{xx} - e^{-2t}\cos 5x = \sum_{n=1}^{\infty}\{\ddot{u}_n + 2\gamma\dot{u}_n + \mu_n^2 u_n - e^{-2t}\delta_{n3}\}\cos\mu_n x$

SINCE THE $\cos\mu_n x$ ARE LINEARLY INDEPENDENT IT FOLLOWS THAT u_n SATISFIES THE ODE

$$\ddot{u}_n + 2\gamma\dot{u}_n + \mu_n^2 u_n = e^{-2t}\delta_{n3}$$

CONSIDER THE HOMOGENEOUS EQ $\ddot{u}_n^H + 2\gamma\dot{u}_n^H + \mu_n^2 u_n^H = 0$

LET $u_n = e^{rt} \Rightarrow r^2 + 2\gamma r + \mu_n^2 = 0 \quad r = \dfrac{-2\gamma \pm \sqrt{4\gamma^2 - 4\mu_n^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \mu_n^2} = -\gamma \pm i\nu_n$

WHERE $\nu_n = \sqrt{\mu_n^2 - \gamma^2} \in \mathbb{R}$ SINCE $\gamma < 1 \leq \mu_n$

THUS $u_n^H = [A_n\cos\nu_n t + B_n\sin\nu_n t]e^{-\gamma t}$

SINCE THE FORCING TERM IS NOT A SOLUTION TO THE HOMOGENEOUS EQ GUESS $u_n^P = Ce^{-2t}$

$\ddot{u}_n^P + 2\gamma u_n^P + \mu_n^2 u_n^P = 4Ce^{-2t} - 4\gamma Ce^{-2t} + \mu_n^2 Ce^{-2t} = C[4 - 4\gamma + \mu_n^2]e^{-2t} = \delta_{n3}e^{-2t}$

$\therefore C_n = \delta_{n3}/[4 - 4\gamma + \mu_n^2] = \delta_{n3}/\beta_n$ WHERE $\beta_n = 4 - 4\gamma + \mu_n^2$

$\therefore u_n(t) = [A_n\cos\nu_n t + B_n\sin\nu_n t]e^{-\gamma t} + \dfrac{\delta_{n3}}{4 - 4\gamma + \mu_n^2}e^{-2t}$

$\therefore u(x,t) = \sum_{n=1}^{\infty}\{e^{-\gamma t}[A_n\cos\nu_n t + B_n\sin\nu_n t] + C_n e^{-2t}\}\cos\mu_n x$

$u_t(x,t) = \sum_{n=1}^{\infty}\{-\gamma e^{-\gamma t}[A_n\cos\nu_n t + B_n\sin\nu_n t] + e^{-\gamma t}[-A_n\nu_n\sin\nu_n t + B_n\nu_n\cos\nu_n t] - 2C_n e^{-2t}\}\cos\mu_n x$

$0 = \sum_{n=1}^{\infty}(A_n + C_n)\cos\mu_n x \Rightarrow A_n = -C_n = -\delta_{n3}/[4 - 4\gamma + \mu_n^2] = -\dfrac{\delta_{n3}}{\beta_n}$

$\cos 3x = \sum_{n=1}^{\infty}\delta_{n2}\cos(2n-1)x = \sum_{n=1}^{\infty}(-\gamma A_n + B_n\nu_n - 2C_n)\cos\mu_n x$

$\therefore B_n = (\delta_{n2} + \gamma A_n + 2C_n)/\nu_n = \dfrac{\delta_{n2}}{\nu_n} - \dfrac{\gamma\delta_{n3}}{(4 - 4\gamma + \mu_n^2)\nu_n} + \dfrac{2\delta_{n3}}{(4 - 4\gamma + \mu_n^2)\nu_n}$

$\therefore u(x,t) = \sum_{n=1}^{\infty}\left\{e^{-\gamma t}\left[-\dfrac{\delta_{n3}}{\beta_n}\cos\nu_n t + \left\{\dfrac{\delta_{n2}}{\nu_n} - \dfrac{\gamma\delta_{n3}}{\beta_n\nu_n} + \dfrac{2\delta_{n3}}{\beta_n\nu_n}\right\}\sin\nu_n t\right] + \dfrac{\delta_{n3}}{\beta_n}e^{-2t}\right\}\cos\mu_n x$

$= \left\{\dfrac{e^{-\gamma t}}{29 - 4\gamma}\left[-\cos\nu_3 t + \dfrac{2-\gamma}{\nu_3}\sin\nu_3 t\right] + \dfrac{e^{-2t}}{29 - 4\gamma}\right\}\cos 5x + \dfrac{e^{-\gamma t}}{\nu_2}\sin\nu_2 t\cos 3x$

4. Consider the following diffusion initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx}, \ 0 < x < \pi/2, \ t > 0 \\ u(0,t) &= 1 - t \text{ and } u_x(\pi/2,t) = 1, \quad t > 0 \\ u(x,0) &= x \end{aligned} \qquad (2)$$

(a) Reduce problem (2) to one with homogeneous boundary conditions and determine the solution to that problem by an eigenfunction expansion.

[14 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution to this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh.

[6 marks]

[total 20 marks]

a) LOOK FOR $W(x,t) = A(t)x + B(t)$ THAT SATISFIES INHOMOGENEOUS BC

$W_x = A(t) = 1 \quad W(0) = B(t) = 1-t$ THUS $\boxed{W(x,t) = x + (1-t)}$ SATISFIES THE BC.

NOW LET $u(x,t) = W(x,t) + V(x,t) \Rightarrow u_t = W_t + V_t = -1 + V_t = u_{xx} = \cancel{W_{xx}} + V_{xx} \Rightarrow \boxed{V_t = V_{xx} + 1}$

BC: $\cancel{1-t} = u(0,t) = W(0,t) + V(0,t) = \cancel{1-t} + V(0,t) \Rightarrow \boxed{V(0,t) = 0}$

$1 = u_x(\tfrac{\pi}{2},t) = W_x(\tfrac{\pi}{2},t) + V_x(\tfrac{\pi}{2},t) = 1 + V_x(\tfrac{\pi}{2},t) \Rightarrow \boxed{V_x(\tfrac{\pi}{2},t) = 0}$

IC: $\cancel{x} = u(x,0) = W(x,0) + V(x,0) = \cancel{x} + 1 + V(x,0) \Rightarrow \boxed{V(x,0) = -1}$

THE EIGENVALUES ASSOCIATED WITH THIS BVP ARE $\mu_n = (2n-1)$ $n = 1, 2, \ldots$ & EIGENFCNS ARE $X_n = \sin(2n-1)x$

LET $S(x) = 1 = \sum_{n=1}^{\infty} S_n \sin\mu_n x \quad S_n = \frac{2}{(\pi/2)} \int_0^{\pi/2} \sin\mu_n x \, dx = \frac{4}{\pi} \cdot \left. \frac{-\cos(2n-1)x}{\mu_n} \right|_0^{\pi/2} = \frac{4}{\pi(2n-1)}$

LET $V(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin\mu_n x \quad V_t = \sum_{n=1}^{\infty} \dot{V}_n \sin\mu_n x \quad V_{xx} = \sum_{n=1}^{\infty} V_n\{-\mu_n^2\} \sin\mu_n x$

$0 = V_t - V_{xx} - 1 = \sum_{n=1}^{\infty} \{\dot{V}_n + \mu_n^2 V_n - S_n\} \sin\mu_n x \xRightarrow{\sin\mu_n x \text{ L.I.}} \dot{V}_n + \mu_n^2 V_n = S_n$

MULTIPLYING BY THE INTEGRATING FACTOR $\frac{d}{dt}[e^{\mu_n^2 t} V_n] = S_n e^{\mu_n^2 t}$

THUS $e^{\mu_n^2 t} V_n = S_n e^{\mu_n^2 t}/\mu_n^2 + C_n \Rightarrow V_n(t) = S_n/\mu_n^2 + C_n e^{-\mu_n^2 t}$

$\therefore V(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{S_n}{\mu_n^2} + C_n e^{-\mu_n^2 t} \right\} \sin\mu_n x$

$-1 = V(x,0) = \sum_{n=1}^{\infty} \left\{ \frac{S_n}{\mu_n^2} + C_n \right\} \sin\mu_n x = \sum_{n=1}^{\infty} b_n \sin\mu_n x$

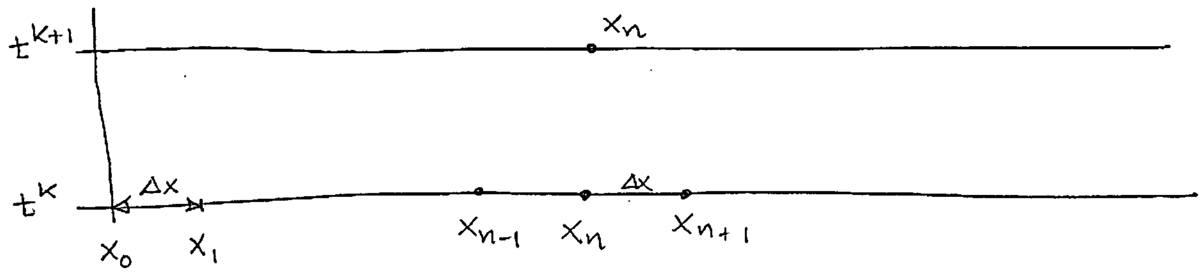
WHERE $b_n = \frac{S_n}{\mu_n^2} + C_n = \frac{2}{\pi/2} \int_0^{\pi/2} (-1) \sin\mu_n x \, dx = -S_n = -\frac{4}{\pi(2n-1)}$

$\therefore \quad C_n = -S_n - \frac{S_n}{\mu_n^2} = -\frac{S_n}{\mu_n^2}(1 + \mu_n^2)$

$\therefore \quad V(x,t) = \sum_{n=1}^{\infty} \frac{S_n}{\mu_n^2}\left[1 - (1+\mu_n^2) e^{-\mu_n^2 t}\right] \sin\mu_n x$

$\therefore \quad u(x,t) = x + (1-t) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left\{ 1 - [1 + (2n-1)^2] e^{-(2n-1)^2 t} \right\} \sin(2n-1)x.$

b) DIVIDE THE SPATIAL DOMAIN $[0, \pi/2]$ INTO N MESHPOINTS $x_n = n\Delta x$ WHERE $\Delta x = \pi/2N$ AND THE TIME INTERVAL $[0,T]$ INTO K TIME STEPS $t^k = k\Delta t$ WHERE $\Delta t = T/K$.



NOW INTRODUCE THE NOTATION $u_n^k = u(x_n, t^k)$ THEN SINCE

$$u_{n\pm1}^k = u(x_n \pm \Delta x, t^k) = u(x_n,t^k) \pm \Delta x\, u_x(x_n,t^k) + \frac{\Delta x^2}{2} u_{xx}(x_n,t^k) \pm \frac{\Delta x^3}{6} u_{xxx}(x_n,t^k) + \frac{\Delta x^4}{24} u_{xxxx}(x_n,t^k) + \cdots$$

$$\therefore\ u_{n+1}^k + u_{n-1}^k = 2u_n^k + \Delta x^2 u_{xx}(x_n,t^k) + O(\Delta x^4)$$

$$\therefore\ \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} = u_{xx}(x_n,t^k) + O(\Delta x^2)$$

$$u_n^{k+1} = u(x_n, t^k + \Delta t) = u(x_n,t^k) + \Delta t\, u_t(x_n,t^k) + \frac{\Delta t^2}{2} u_{tt}(x_n,t^k) + \cdots$$

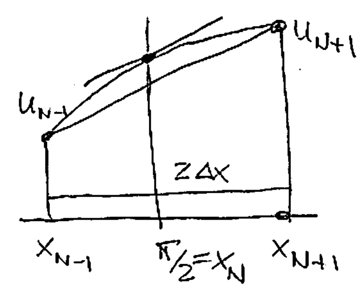
$$\therefore\ \frac{u_n^{k+1} - u_n^k}{\Delta t} = u_t(x_n,t^k) + O(\Delta t)$$

THUS REPLACING THE DERIVATIVES IN $u_t = u_{xx}$ BY DIFFERENCES WE OBTAIN

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} + O(\Delta t, \Delta x^2)$$

OR $$u_n^{k+1} = u_n^k + \frac{\Delta t}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$$

TO APPROXIMATE THE DERIVATIVE BC WE USE THE $O(\Delta x^2)$ CENTRAL DIFFERENCE APPROXIMATION



$$u_x(x_N, t) = \frac{u_{N+1} - u_{N-1}}{2\Delta x} + O(\Delta x^2) = 1$$

OR $u_{N+1} = u_{N-1} + 2\Delta x$

5. Consider the following eigenvalue problem

$$
\begin{aligned}
Ly \quad x^2y'' + 3xy' + \lambda y &= 0,\ 1 < x < e^\pi \qquad (1)\\
y(1) &= 0 \text{ and } y(e^\pi) = 0 \qquad (2)
\end{aligned}
$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions (consider the cases $\lambda < 1,\ \lambda = 1,\ \lambda > 1$ separately).

(b) Now use the result in part (a) to solve the following heat conduction problem by separation of varaibles:

$$
\begin{aligned}
u_t &= x^2u_{xx} + 3xu_x,\ 1 < x < e^\pi,\ t > 0\\
u(1,t) &= 0, \text{ and } u(e^\pi,t) = 0\\
u(x,0) &= 2/x.
\end{aligned}
$$

[total 20 marks]

a) LET $F = e^{\int \frac{3x}{x^2}dx}/x^2 = e^{\ln x^3}/x^2 = x$ SO MULTIPLY (1) BY F TO OBTAIN

$x^3y'' + 3x^2y' + \lambda x y = 0$ OR $-(x^3y')' = \lambda x y$ WHICH IS IN S-L FORM

SINCE (1) IS A C-E EQ LET $y = x^\gamma \Rightarrow \gamma(\gamma-1) + 3\gamma + \lambda = \gamma^2 + 2\gamma + \lambda = 0$

$\therefore \gamma = \frac{-2 \pm \sqrt{4-4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}$

$\lambda<1$: $\gamma = -1 \pm \sqrt{1-\lambda} = -1 \pm \theta \Rightarrow y = x^{-1}[A\cosh\theta\ln x + B\sinh\theta\ln x]$, $0 = y(1) = A$ $\Big\}$ $y \equiv 0$ TRIVIAL

$0 = y(e^\pi) = B\sinh\theta\pi \overset{\theta>0}{\Rightarrow} B = 0$

$\lambda=1$: $\gamma = -1$ IS A DOUBLE ROOT $y = x^{-1}[A + B\ln x]$ $0 = y(1) = A$ $0 = y(e^\pi) = e^{-\pi}B.\pi \Rightarrow B = 0$ TRIVIAL

$\lambda>1$: $\gamma = -1 \pm i\sqrt{\lambda-1} = -1 \pm i\mu$ WHERE $\mu = \sqrt{\lambda-1} \Rightarrow y(x) = x^{-1}[A\cos\mu\ln x + B\sin\mu\ln x]$

$0 = y(1) = A$ $0 = y(\pi) = Be^{-\pi}\sin\mu\ln e^\pi = Be^{-\pi}\sin\mu\pi \Rightarrow \boxed{\mu_n = n \ \ n = 1,2,\ldots \ \ y_n = \sin(n\ln x)/x}$

$\lambda_n = 1 + n^2 \quad n = 1,2,\ldots$

$y_n = x^{-1}\sin(n\ln x)$

b) LET $u(x,t) = X(x)T(t) \Rightarrow X\dot{T} = (x^2X'' + 3xX')T$

$\div XT]$ $\frac{\dot{T}}{T(t)} = \frac{x^2X'' + 3xX'}{X(x)} = -\lambda$

$T]$ $\dot{T} = -\lambda T \Rightarrow T(t) = ce^{-\lambda t}$

$X]$ $x^2X'' + 3xX' + \lambda X = 0$ $X(1) = 0 = X(e^\pi)$

THUS $\lambda_n = (1+n^2)$ $n = 1,2..$ ARE THE EIGENVALUES AND $X_n = x^{-1}\sin(n\ln x)$ THE EIGENFUNCTIONS

$\therefore u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(1+n^2)t} x^{-1}\sin(n\ln x)$ LET $\chi = \ln x$ $d\chi = \frac{dx}{x}$

$2/x = u(x,0) = \sum_{n=1}^{\infty} b_n x^{-1}\sin(n\ln x)$ $x = 1 \Rightarrow \chi = 0$ $x = e^\pi \Rightarrow \chi = \pi$

$\int_1^{e^\pi} \cancel{x}\left(\frac{2}{\cancel{x}}\right)x^{-1}\sin(m\ln x)dx = \sum_{n=1}^{\infty} b_n \int_1^{e^\pi} x.x^{-1}\sin(m\ln x).x^{-1}\sin(n\ln x)\,dx$

$\therefore 2\int_0^\pi \sin m\chi\, d\chi = \sum_{n=1}^{\infty} b_n \int_0^\pi \sin(m\chi)\sin(n\chi)d\chi = \sum_{n=1}^{\infty} b_n\delta_{mn}\pi/2$

$\therefore -2\frac{\cos m\chi}{m}\Big|_0^\pi = \frac{2[1+(-1)^{m+1}]}{m} = b_m\pi/2 \Rightarrow b_m = \frac{4}{\pi}\frac{[1+(-1)^{m+1}]}{m}$

$\therefore u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{[1+(-1)^{n+1}]}{n} e^{-(1+n^2)t} x^{-1}\sin(n\ln x)$