1. $Ly = (x^2-1)y'' + xy' - y = 0$

(a) $P(x) = x^2 - 1$, $Q(x) = x$, $R(x) = -1$

$P(x) = 0 \rightarrow x = \pm 1$ singular points

$x \neq \pm 1$ ordinary points

$$\lim_{x\to 1} \frac{x}{(x^2-1)}(x-1) = \lim_{x\to 1} \frac{x}{(x+1)} = \frac{1}{2} \quad \text{finite}$$

$$\lim_{x\to 1} \frac{-1}{(x^2-1)}(x-1)^2 = \lim_{x\to 1} \frac{-(x-1)}{(x+1)} = 0 \quad \text{finite}$$

$x = 1$ is a R.S.P.

$$\lim_{x\to -1} \frac{x}{(x^2-1)}(x+1) = \lim_{x\to -1} \frac{x}{(x-1)} = \frac{+1}{+2} \quad \text{finite}$$

$$\lim_{x\to -1} \frac{-(x+1)^2}{(x^2-1)} = \lim_{x\to -1} -\frac{(x+1)}{(x-1)} = 0 \quad \text{finite}$$

$x = -1$ is a R.S.P.

(b) $x = 0$: ordinary point

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) \cdot x^{n-2}$$

$$Ly = x^2 y'' - y'' + x y' - y = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^n - \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$\boxed{m=n}$ $\boxed{\begin{array}{l} m = n-2 \\ n=2 \rightarrow m=0 \\ n = m+2 \end{array}}$

$$+ \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=1}^{\infty} a_n x^n = 0$$

$\boxed{m=n}$ $\boxed{m=n}$

→ Change the indices:

$$\sum_{m=2}^{\infty} a_m m(m-1) x^m - \sum_{m=0}^{\infty} a_{m+2} (m+2)(m+1) x^m$$

$$+ \sum_{m=1}^{\infty} a_m m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

Start all indices from $m=2$:

$$\sum_{m=2}^{\infty} a_m m(m-1) x^m - a_2 \, 2 \cdot 1 \cdot x^0 - a_3 \cdot 3 \cdot 2 \cdot x$$

$$- \sum_{m=2}^{\infty} a_{m+2} (m+2)(m+1) x^m + a_1 x^1 + \sum_{m=2}^{\infty} a_m m x^m$$

$$- a_0 \cdot x^0 - a_1 x - \sum_{m=2}^{\infty} a_m x^m = 0$$

Set the Coefficients to zero :

$$x^0] \quad -2a_2 - a_0 = 0 \longrightarrow a_2 = -\frac{a_0}{2}$$

$$x^1] \quad -6a_3 + a_1 - a_1 = 0 \longrightarrow a_3 = 0$$

$$x^m] \quad -a_{m+2}(m+2)(m+1) + a_m(m^2 \cancel{- m + m} - 1) = 0$$

$m \geq 2$

$$a_{m+2} = \frac{a_m(m^2-1)}{(m+2)(m+1)} = \frac{a_m(m-1)}{(m+2)}$$

$$m=2: \quad a_4 = \frac{a_2 \cdot 1}{4} = +\frac{a_0}{8}$$

$$m=3: \quad a_5 = \frac{a_3 \cdot 2}{5} = 0 = a_7 = a_9 = \cdots$$

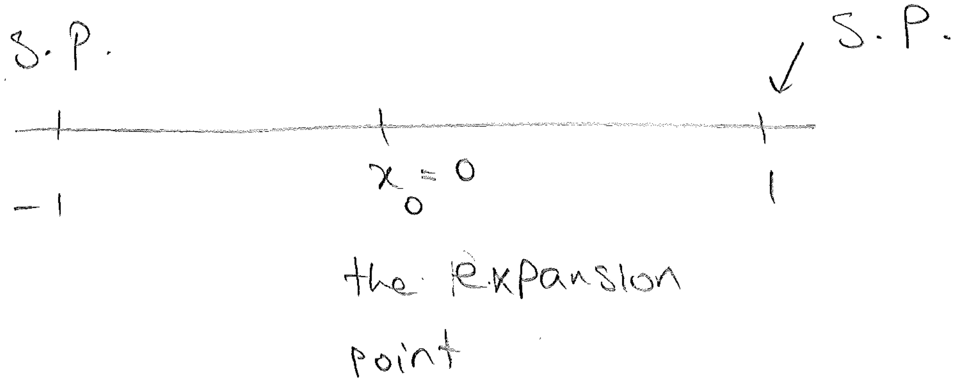
$$m=4: \quad a_6 = \frac{a_4 \cdot 3}{6} = \frac{-a_0 \cdot 3}{6 \cdot 8} = -\frac{a_0}{16}$$

$$m=6: \quad a_8 = \frac{a_6 \cdot 5}{8} = \frac{-a_0 \cdot 5}{16 \cdot 8} = -\frac{5a_0}{128}$$

a_1 is arbitrary :

$$y(x) = a_1 x + a_0\left[1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} \cdots\right]$$

(c)



$\rho = 1$: the distance from x_0 to the nearest singular point.

Or, do the ratio test:

$$\lim_{m \to \infty} \left| \frac{1-m}{m+2} x^2 \right| < 1$$

$$|x^2| \underbrace{\lim_{m \to \infty} \left| \frac{1-m}{m+2} \right|}_{1} < 1 \quad \to \quad |x^2| < 1$$

$$-1 < x < 1$$

$$\rho = 1$$

problem 2:

$$u_t = u_{xx} + \sin(x) \,, \qquad 0 < x < \pi \,, \ t > 0$$

$$u(0,t) = 1, \quad u(\pi,t) = 0$$

$$u(x,0) = -\frac{x}{\pi}$$

(a) $u_\infty(x) = w(x)$

$$w_{xx} + \sin x = 0 \rightarrow w_{xx} = -\sin x$$

$$w_x = \cos x + C_1$$

$$w = \sin x + C_1 x + C_2$$

$$w(0) = 1 \rightarrow C_2 = 1, \quad w(\pi) = 0 = C_1 . \pi + 1 \rightarrow C_1 = -\frac{1}{\pi}$$

$$w(x) = \sin x - \frac{x}{\pi} + 1$$

(b) $u(x,t) = w(x) + v(x,t)$

PDE: $u_t = u_{xx} + \sin x$

$$\underbrace{\cancel{w_t}}_{0} + v_t = w_{xx} + v_{xx} + \sin x$$

$$v_t = -\cancel{\sin x} + v_{xx} + \cancel{\sin x} \rightarrow v_t = v_{xx}$$

BC: $u(0,t) = 1 = w(0) + v(0,t) = 1 + v(0,t) \rightarrow v(0,t) = 0$

$u(\pi,t) = 0 = w(\pi) + v(\pi,t) = 0 + v(\pi,t) \rightarrow v(\pi,t) = 0$

IC: $u(x,0) = \cancel{-\frac{x}{\pi}} = w(x) + v(x,0) = \sin x - \cancel{\frac{x}{\pi}} + 1 + v(x,0)$

$$v(x,0) = -\sin x - 1$$

(c) $v_t = v_{xx}$

$v(0,t) = v(\pi,t) = 0$

$v(x,0) = -\sin x - 1$

$$V(x,t) = X(x) \cdot T(t)$$

$$\dot{T} X = T X'' \rightarrow \frac{\dot{T}}{T} = \frac{X''}{X} = \lambda$$

$$\left.\begin{array}{l} X'' - \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{array}\right\} \rightarrow \begin{array}{l} X_n = \sin(nx) \\ \lambda_n = -n^2 \end{array}$$

$$\frac{\dot{T}}{T} = -n^2 \rightarrow T = e^{-n^2 t}$$

$$V(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-n^2 t}$$

$$V(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx) = -\sin x - 1$$

$$b_n = -\frac{2}{\pi} \int_0^{\pi} \sin x \cdot \sin(nx)\, dx - \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx$$

$$= -\delta_{n1} + \frac{2}{\pi} \cdot \frac{1}{n} \cos(nx) \Big|_0^{\pi}$$

$$b_n = -\delta_{n1} + \frac{2}{n\pi} \left[(-1)^n - 1 \right]$$

$$V(x,t) = -\sin x\, e^{-t} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[(-1)^n - 1 \right] \sin(nx)\, e^{-n^2 t}$$

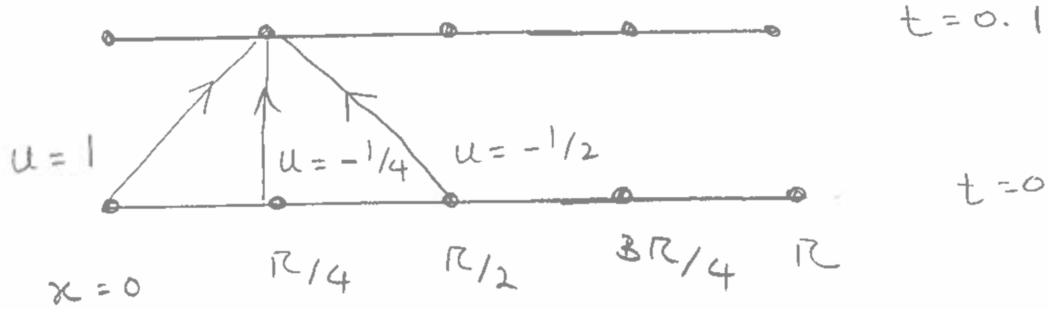
$$u(x,t) = \sin x - \frac{x}{\pi} + 1 + V(x,t)$$

(d)

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} =$$

$$\frac{u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)}{\Delta x^2} + \sin(x)$$

$\Delta x = \pi/4$, $\Delta t = 0.1$



$$u(x,t+\Delta t) = u(x,t) + \frac{\Delta t}{\Delta x^2}\left[u(x+\Delta x,t)-2u(x,t)+u(x-\Delta x,t)\right)$$

$$+ \Delta t \sin x$$

$$u\left(\frac{\pi}{4},0.1\right) = -\frac{1}{4} + \frac{0.1}{(\pi/4)^2}\left[\cancel{-\frac{1}{2}} + \cancel{2\cdot\frac{1}{4}} + 1\right] + 0.1\sin(\pi/4)$$

$$= -\frac{1}{4} + (0.1)\frac{16}{\pi^2}[1] + 0.1\frac{\sqrt{2}}{2}$$

$$= -\frac{1}{4} + \frac{1.6}{\pi^2} + 0.1\frac{\sqrt{2}}{2}$$

Problem 3:

$$u_{tt} + \gamma u_t = u_{xx} + e^{-2t} \sin\left(\frac{3x}{2}\right) \qquad 0 < x < \pi$$

$$u(0,t) = 0, \quad u_x(\pi, t) = 0$$

$$u(x,0) = \sin\left(\frac{5x}{2}\right), \quad u_t(x,0) = 0$$

Method of eigenfunction expansions :

$$X_n = \sin\left(\frac{(2n-1)\cancel{\pi} x}{2\cancel{\pi}}\right) = \sin\left(\frac{(2n-1)x}{2}\right)$$

$$\mu_n = \frac{2n-1}{2}, \qquad n = 1, 2, 3, \dots$$

$$S(x,t) = e^{-2t} \sin\left(\frac{3x}{2}\right) = \sum_{n=1}^{\infty} S_n(t) \sin\left(\frac{(2n-1)x}{2}\right)$$

$$S_n(t) = e^{-2t} \delta_{2n}$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) . \sin(\mu_n x)$$

$$u_t = \sum_{n=1}^{\infty} u_n'(t) . \sin(\mu_n x)$$

$$u_{tt} = \sum_{n=1}^{\infty} u_n''(t) . \sin(\mu_n x)$$

$$u_{xx} = \sum_{n=1}^{\infty} u_n(t) . (-\mu_n^2) \sin(\mu_n x)$$

$$u_{tt} + \gamma u_t - u_{xx} - e^{-2t} \sin\left(\frac{3x}{2}\right) = 0$$

$$\sum_{n=1}^{\infty} \left\{ u_n''(t) + \gamma u_n'(t) + \mu_n^2 u_n(t) - e^{-2t} \delta_{n2} \right\} \sin(\mu_n x) = 0$$

$$\rightarrow \quad u_n''(t) + \gamma u_n'(t) + \mu_n^2 u_n(t) = e^{-2t} \delta_{n2} \quad (*)$$

Find the complementry solution to the homogeneous ODE:

$$u_n^c(t) = e^{rt} \rightarrow r^2 + \gamma r + \mu_n^2 = 0 \rightarrow r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\mu_n^2}}{2}$$

$$0 < \gamma < 1 \quad , \quad \mu_n = \frac{2n-1}{2} \quad , \quad n = 1 \; , \; \mu_1 = \frac{1}{2}$$ the smallest μ_n

So $\gamma^2 < 4\mu_1^2 \rightarrow r = -\frac{\gamma}{2} \pm i\alpha_n \quad , \alpha_n = \sqrt{4\mu_n^2 - \gamma^2}$

$$u_n^c(t) = \left[A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right] e^{-\gamma/2\, t}$$

Now, find the particular solution: guess: $u_n^p(t) = C_n e^{-2t}$

$$\left[4 \cdot C_n - 2\gamma C_n + \mu_n^2 C_n\right] e^{-2t} = e^{-2t} \delta_{n2}$$

$$\rightarrow \quad C_n = \frac{\delta_{n2}}{(4 - 2\gamma + \mu_n^2)}$$

$$\rightarrow \quad u_n(t) = \left[A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t)\right] e^{-\gamma/2\, t} + \frac{\delta_{n2} \, e^{-2t}}{(4 - 2\gamma + \mu_n^2)}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[u_n(t) \right] \cdot \sin(\mu_n x)$$

Finding A_n's and B_n's :

$$u(x,0) = \sin\left(\frac{5x}{2}\right) = \sum_{n=1}^{\infty} \left[A_n + \frac{\delta_{n2}}{(4-2\gamma+\mu_n^2)} \right] \sin(\mu_n x)$$

$$\rightarrow \quad A_n = \delta_{n3} - \frac{\delta_{n2}}{4-2\gamma+\mu_n^2}$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \Big\{ \left[A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t) \right] \left(-\frac{\gamma}{2} e^{-\gamma/2\, t} \right)$$

$$+ \left[-A_n \alpha_n \sin(\alpha_n t) + B_n \alpha_n \cos(\alpha_n t) \right] e^{-\gamma/2\, t}$$

$$- \frac{2\delta_{n2} e^{-2t}}{4-2\gamma+\mu_n^2} \Big\} . \sin(\mu_n x)$$

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} \left\{ A_n . (-\gamma/2) + B_n \alpha_n - \frac{2\delta_{n2}}{4-2\gamma+\mu_n^2} \right\} . \sin(\mu_n x$$

$$\rightarrow \quad B_n = \frac{1}{\alpha_n} \left\{ A_n . \frac{\gamma}{2} + \frac{2\delta_{n2}}{4-2\gamma+\mu_n^2} \right\}$$

$$u(x,t) = \sum_{n=1}^{\infty} \Big\{ \Big[\left(\delta_{n3} - \frac{\delta_{n2}}{4-2\gamma+\mu_n^2} \right) \cos(\mu_n t)$$

$$+ \frac{1}{\alpha_n} \left(\delta_{n3} . \frac{\gamma}{2} - \frac{\delta_{n2} . \gamma}{2(4.2\gamma+\mu_n^2)} + \frac{2\delta_{n2}}{4-2\gamma+\mu_n^2} \right) \sin(\mu_n t) \Big] . e^{-\gamma/2\, t}$$

$$+ \frac{\delta_{n2} e^{-2t}}{(4-2\gamma+\mu_n^2} \Big\}$$

$$u(x,t) = \cos\left(\frac{5x}{2}\right) \cdot e^{-\gamma/2\, t} - \frac{\cos(3x/2)}{4-2\gamma+\left(\frac{3}{2}\right)^2} \cdot e^{-\gamma/2\, t}$$

$$+ \frac{1}{\sqrt{4 \cdot \frac{5^2}{2^2} - \gamma^2}} \, \frac{\gamma}{2} \cdot \sin\left(\frac{5x}{2}\right) e^{-\gamma/2\, t} - \frac{\gamma \cdot \sin\left(\frac{3x}{2}\right) e^{-\gamma/2\, t}}{2\left(4 \cdot 2\gamma + \left(\frac{3}{2}\right)^2\right)} \cdot \frac{1}{\sqrt{4 \cdot \frac{9}{4} - \gamma^2}}$$

$$+ \frac{2}{4 - 2\gamma + \left(\frac{3}{2}\right)^2} \sin\left(\frac{3}{2} t\right) \cdot e^{-\gamma/2\, t} \cdot \frac{1}{\sqrt{9-\gamma^2}}$$

$$+ \frac{e^{-2t}}{4 - 2\gamma + \left(\frac{3}{2}\right)^2}$$

(b) If $1 < \gamma < 2$, then when solving the ODE's in $(*)$:

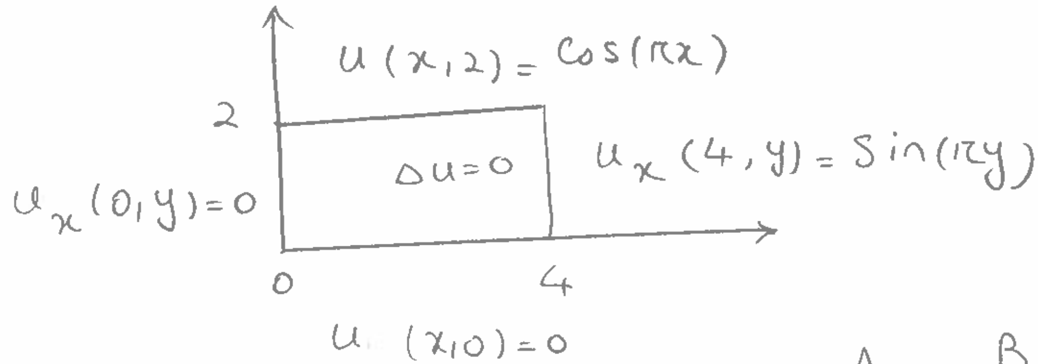
$\gamma^2 - 4\mu_1^2 = \gamma^2 - 1 > 0 \rightarrow$ r_1, r_2 are two real negative roots.

$\gamma^2 - 4\mu_2^2 = \gamma^2 - 4 \cdot \left(\frac{3}{2}\right)^2 = \gamma^2 - 9 < 0 \rightarrow$ the same as before, two complex roots.

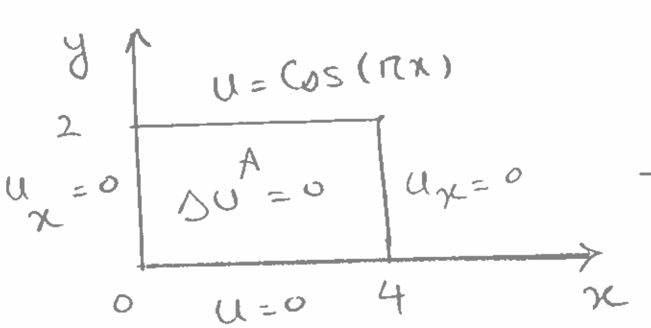
So, the solution to the first mode will be different:

$$u_1^c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad (r_1, r_2 < 0)$$

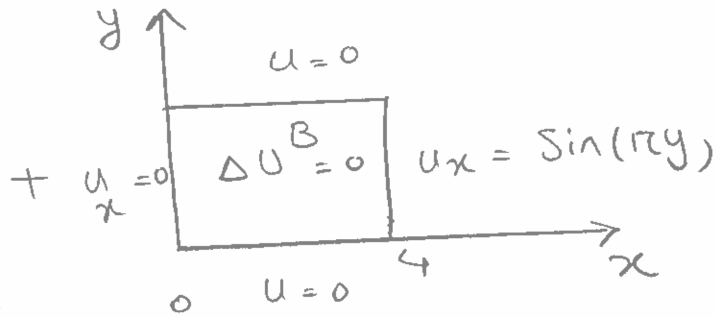
problem 4:



Divide it to 2 subproblems: $\Delta u = \Delta u^A + \Delta u^B = 0$



problem A
eigenvalue problem in x



problem B
eigenvalue problem in y

$u(x,y) = X(x) \cdot Y(y)$

$u_{xx} + u_{yy} = 0 \rightarrow X''Y + Y''X = 0 \rightarrow \dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda$

problem A:

$$\left.\begin{aligned} & X'' - \lambda X = 0 \\ & X'(0) = 0 = X'(4) \end{aligned}\right\} \text{Neumann problem}$$

$\lambda_0 = 0, \quad X_0 = 1$

$\lambda_n = -\mu_n^2, \quad X_n = \cos(\mu_n x)$

$\mu_n = \dfrac{n\pi}{4}, \quad n = 1, 2, \ldots$

$Y'' + \lambda Y = 0$

$\lambda_0 = 0: \quad Y'' = 0 \rightarrow Y_0 = A_0 + B_0 y$

$Y_0(0) = 0 \rightarrow A_0 = 0$

$\lambda_n = -\mu_n^2: \quad Y_n(y) = A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y)$

$$Y_n(0) = 0 \rightarrow A_n \cdot 1 + B_n \cdot 0 = 0 \rightarrow A_n = 0$$

$$u^A(x,y) = B_0 y + \sum_{n=1}^{\infty} B_n \sinh(\mu_n y) \cos(\mu_n x)$$

$$u^A(x,2) = \cos(\pi x) = B_0 \cdot 2 + \sum_{n=1}^{\infty} B_n \sinh(2\mu_n) \cos(\mu_n x)$$

$$\mu_n = \frac{n\pi}{4}, \quad B_0 = 0, \quad B_n \sinh(2\mu_n) = \delta_{n4}$$

$$u^A(x,y) = \sum_{n=1}^{\infty} \delta_{n4} \cdot \frac{\sinh(\mu_n y)}{\sinh(2\mu_n)} \cos(\mu_n x) \qquad n=4 \rightarrow \mu_4 = \pi$$

$$u^A(x,y) = \frac{\sinh(\pi y)}{\sinh(2\pi)} \cdot \cos(\pi x)$$

Problem B:

$$\left.\begin{array}{l} Y'' + \lambda Y = 0 \\ Y(0) = 0 = Y(2) \end{array}\right\} \text{Dirichlet problem}$$

$$Y_n = \sin\left(\frac{n\pi y}{2}\right)$$

$$\lambda_n = \mu_n^2, \quad \mu_n = \frac{n\pi}{2}, \quad n = 1, 2, 3, \ldots$$

$$X'' - \mu_n^2 X = 0 \rightarrow X_n = A_n \cosh(\mu_n x) + B_n \sinh(\mu_n x)$$

$$X'_n = A_n \mu_n \sinh(\mu_n x) + B_n \mu_n \cosh(\mu_n x)$$

$$X'_n(0) = 0 = 0 + B_n \cdot \mu_n \cdot 1 \rightarrow B_n = 0$$

$$u^B(x,y) = \sum_{n=1}^{\infty} A_n \cosh(\mu_n x) \cdot \sin\left(\frac{n\pi y}{2}\right)$$

$$u^B_x(x,y) = \sum_{n=1}^{\infty} A_n \mu_n \sinh(\mu_n x) \sin\left(\frac{n\pi y}{2}\right)$$

$$u_x^B(4,y) = \sin(\pi y) = \sum_{n=1}^{\infty} \underbrace{A_n \mu_n \sinh(\mu_n 4)}_{D_n} \sin\left(\frac{n\pi y}{2}\right)$$

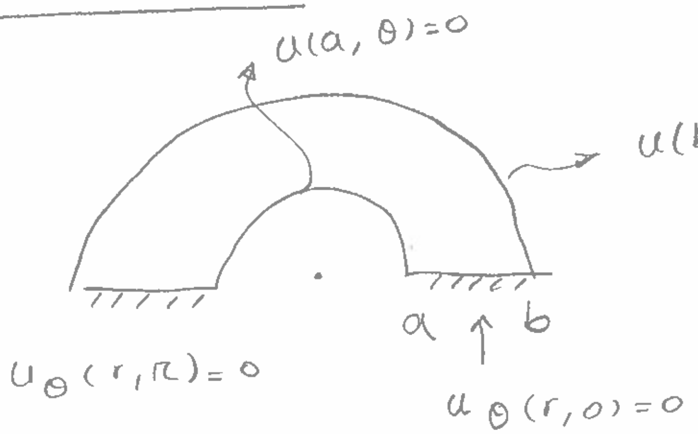
$$D_n = \delta_{n2} \quad , \quad n=2 \rightarrow \mu_n = \pi$$

$$A_2 = \frac{1}{\pi \cdot \sinh(4\pi)}$$

$$u^B(x,y) = \frac{1}{\pi} \frac{\cosh(\pi x)}{\sinh(4\pi)} \sin(\pi y)$$

$$u(x,y) = u^A(x,y) + u^B(x,y)$$

problem 5:



$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$u(r,\theta) = R(r) \cdot \Theta(\theta)$$

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R \cdot \Theta'' = 0 \longrightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$\left.\begin{array}{l} \Theta'' + \lambda\Theta = 0 \\ \Theta'(0) = \Theta'(\pi) = 0 \end{array}\right\} \quad \begin{array}{l} \lambda_0 = 0 \ , \quad \Theta_0 = 1 \\ \lambda_0 = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \quad X_0 = \cos(n\theta) \end{array}$$

$$r^2 R'' + rR' - \lambda R = 0$$

$$\lambda_0 = 0 \longrightarrow r^2 R'' + rR' = 0, \qquad R = x^r, R' = r x^{r-1}, R'' = (r-1) r x^{r-2}$$

$$r(r-1)x^r + r x^r = 0 \longrightarrow r = 0, 0$$

$$R_0(r) = A_0 + B_0 \ln r$$

$$\lambda_n = n^2 \longrightarrow \quad r^2 R'' + rR' - n^2 R = 0 \quad , \; R = x^r$$

$$r(r-1)x^r + r x^r - n^2 x^r = 0$$

$$r^2 - r + r - n^2 = 0 \longrightarrow r = \pm n$$

$$R_n(r) = A_n r^n + B_n r^{-n}$$

$$u(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos(n\theta)$$

$$u(a,\theta) = 0 \rightarrow A_0 + B_0 \ln a = 0 \rightarrow A_0 = -B_0 \ln a$$

$$A_n a^n + B_n a^{-n} = 0 \rightarrow B_n = -A_n a^{2n}$$

$$u(r,\theta) = B_0 \ln(r/a) + \sum_{n=1}^{\infty} A_n (r^n \mp a^{2n} r^{-n}) \cos(n\theta)$$

$$u(b,\theta) = B_0 \ln(b/a) + \sum_{n=1}^{\infty} A_n (b^n \mp a^{2n} b^{-n}) \cos(n\theta)$$

$$= 1 + \cos\theta$$

$$B_0 = \frac{1}{\ln b/a}, \quad A_n (b^n \mp a^{2n} b^{-n}) = \delta_{n1} \rightarrow A_n = \frac{\delta_{n1}}{b^n \pm a^{2n} b^{-2}}$$

$$u(r,\theta) = \frac{\ln(r/a)}{\ln(b/a)} + \frac{\left(\left(\frac{r}{a}\right)^n \mp \left(\frac{r}{a}\right)^{-n} \right) \cos(\theta)}{\left(\left(\frac{b}{a}\right)^n \pm \left(\frac{b}{a}\right)^{-n} \right)}$$