Math 257/316 : Final Exam

April 19, 2023, Time: 2.5 hours

- The test consists of 26 pages and 5 questions worth a total of 100 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.) A formula sheet is provided.

Student number								
Section								
Name	....							
Signature								

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. Consider the second order differential equation

$$Ly = (x^2 - 1)y'' + xy' - y = 0, \tag{1}$$

(a) Classify the points $-\infty < x < \infty$ as ordinary points, regular singular points and irregular singular points.

[3 marks]

(b) Use the appropriate series expansion about the point $x = 0$ to determine two independent solutions to equation (1) . You only need to determine the first three non-zero terms in each case.

[15 marks]

(c) What can you say about the minimum radius of convergence of the series solution you found in (b)?

[2 marks]
[total 20 marks]

Question 1 (continued):

Question 1 (continued):

Question 1 (continued):

Question 1 (continued):

2. Consider the following boundary value problem for the heat equation governing the temperature within a conducting bar:

$$u_t = u_{xx} + \sin(x), \quad 0 < x < \pi, \quad t > 0 \tag{2}$$

$$BC : u(0,t) = 1, \quad u(\pi,t) = 0 \tag{3}$$

$$IC : u(x,0) = \frac{-x}{\pi} \tag{4}$$

(a) Determine the steady-state temperature $u_\infty(x)$.

[3 marks]

(b) Let $u(x,t) = u_\infty(x) + v(x,t)$ and identify the PDE, BC and IC satisfied by $v(x,t)$.

[3 marks]

(c) Use the method of separation of variables to solve the above boundary value problem for $v(x,t)$ and from this determine the solution $u(x,t)$.

[9 marks]

(d) Consider the following discretized equation that uses the finite differences approximations to solve the PDE in equation (2):

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} + \sin(x) \tag{5}$$

Assuming a time step size of $\Delta t = 0.1$ and a spatial mesh size of $\Delta x = \pi/4$, use equation (5) to estimate the temperature of the bar at $x = \pi/4$ and $t = 0.1$, i.e., $u(\pi/4, 0.1)$.

[5 marks]

[total 20 marks]

Question 2 (continued):

Question 2 (continued):

Question 2 (continued):

Question 2 (continued):

3. Consider the following initial-boundary value problem for the wave equation with damping

$$u_{tt} + \gamma u_t = u_{xx} + e^{-2t} \sin(\frac{3x}{2}) \qquad t > 0, \quad 0 \leq x \leq \pi \tag{6}$$

$$u(0, t) = 0, \quad \text{and} \quad u_x(\pi, t) = 0 \tag{7}$$

$$u(x, 0) = \sin(\frac{5x}{2}), \quad \text{and} \quad u_t(x, 0) = 0 \tag{8}$$

Assuming that the damping coefficient $0 < \gamma < 1$, use the method of eigenfunction expansions to determine the solution to this problem.

[25 marks]

Question 3 (continued):

Question 3 (continued):

Question 3 (continued):

Question 3 (continued):

Question 3 (continued):

4. Consider the Laplace's equation on a rectangular domain subject to the following boundary conditions that represents the steady-state heating of a plate.

$$u_{xx} + u_{yy} = 0 \qquad 0 \le x \le 4 \text{ and } 0 \le y \le 2 \tag{9}$$

$$u_x(0, y) = 0, \;\; u_x(4, y) = \sin(\pi y), \;\; u(x, 0) = 0, \;\; u(x, 2) = \cos(\pi x). \tag{10}$$

Solve this problem using the method of separation of variables.

[total 20 marks]

Question 4 (continued):

Question 4 (continued):

Question 4 (continued):

Question 4 (continued):

5. Use the method of separation of variables to solve the following boundary value problem for an annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad a < r < b, \quad 0 < \theta < \pi \quad (11)$$

$$u_\theta(r,0) = 0, \quad u_\theta(r,\pi) = 0, \quad u(a,\theta) = 0 \quad \text{and} \quad u(b,\theta) = \cos(\theta) + 1 \quad (12)$$

[total 15 marks]

Question 5 (continued):

Question 5 (continued):

Question 5 (continued):