# Midterm 1, Section 202, Solutions:

## Question 1:

$$\mathcal{L}y = 3x^2 y'' + 2x(x+1)y' - 2y = 0$$

(a) $P(x) = 3x^2$, $Q(x) = 2x(x+1)$, $R(x) = -2$

$$\frac{Q(x)}{P(x)} = \frac{2x(x+1)}{3x^2} = \frac{2(x+1)}{3x}, \quad \frac{R(x)}{P(x)} = \frac{-2}{3x^2}$$

$x=0$: singular point, all other points: ordinary points

$$\lim_{x\to 0} x \cdot \frac{2x(x+1)}{3x^2} = \frac{2}{3} \text{ finite}, \quad \lim_{x\to 0} \frac{x^2(-2)}{3x^2} = -\frac{2}{3} \text{ Finite}$$

So $x=0$ is a R. S. P.

(b) Assume a Frobenius Series Solution:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$\mathcal{L}y = 3x^2 y'' + 2x^2 y' + 2xy' - 2y = 0$$

$$Ly = 3\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r} + 2\sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r+1}$$

$\boxed{m=n}$ $\boxed{\begin{array}{l} m+r = n+r+1 \\ m = n+1,\ n=m-1 \\ n=0 \to m=1 \end{array}}$

$$+\, 2\sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r} - 2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$\boxed{m=n}$ $\boxed{m=n}$

Shift of index and peel-off:

$$[3a_0 r(r-1) + 2a_0 r - 2a_0]x^r$$

$$+ \sum_{m=1}^{\infty} \left[ 3a_m (m+r)(m+r-1) + 2a_{m-1}(m+r-1) + 2a_m (m+r) - 2a_m \right] x^{m+r} = 0$$

$\underline{x^r}$: $\quad a_0 [3r^2 - 3r + 2r - 2] = 0 \to 3r^2 - r - 2 = 0$

$$r_{1,2} = \frac{1 \pm \sqrt{1+24}}{6} = 1, -2/3$$

$\dfrac{x^{m+r}}{(m \geq 1)}$: $\quad a_m [3(m+r)(m+r-1) + 2(m+r) - 2] = -2a_{m-1}(m+r-1)$

$$a_m [(m+r)(3m+3r-3+2) - 2] = -2a_{m-1}(m+r-1)$$

$$a_m = \frac{-2a_{m-1}\,(m+r-1)}{(m+r)[3(m+r)-1]-2}$$

$\underline{r_1 = 1}$: $\quad a_m = \dfrac{-2a_{m-1} \cdot m}{(m+1)(3m+2)-2}$

$m=1$: $\quad a_1 = \dfrac{-2a_0}{2.5-2} = -\dfrac{a_0}{4}$

$m=2$: $\quad a_2 = \dfrac{-2a_1 . 2}{3.8-2} = -\dfrac{2a_1}{11} = \dfrac{a_0}{22}$

$$y_1(x) = a_0 x^1 \left[ 1 - \frac{x}{4} + \frac{x^2}{22} + \cdots \right]$$

$\underline{r = -2/3:}$ $$a_m = \frac{-2a_{m-1}\,(m-5/3)}{(m-\frac{2}{3})\,[3m-2-1]-2}$$

$$a_m = \frac{-2a_{m-1}\,(3m-5)}{(3m-2)(3m-3)-6}$$

$m=1:$ $$a_1 = \frac{-2a_0(-2)}{1\cdot 0-6} = -\frac{2}{3}a_0$$

$m=2:$ $$a_2 = \frac{-2\,a_1\cdot 1}{4\cdot 3-6} = \frac{-2a_1}{6} = \frac{2a_0}{9}$$

$$y_2(x) = a_0\,x^{-2/3}\left[1-\frac{2}{3}x+\frac{2}{9}x^2+\cdots\right]$$

## Question 2 :

$y'' + \lambda y = 0 \quad (0 < x < \pi/2), \quad y(0) = 0, \quad y(\pi/2) = 0$

guess Solution: $y(x) = e^{rx} \rightarrow [r^2 + \lambda]e^{rx} = 0, \quad r^2 = -\lambda$

Case 1: $\lambda > 0$ or $\lambda = \mu^2 \rightarrow \quad r = \pm i\mu$

$y(x) = A\cos(\mu x) + B\sin(\mu x)$

$y(0) = 0 \rightarrow \quad A \cdot 1 + 0 = 0, \quad A = 0$

$y(\pi/2) = 0 \rightarrow \quad B \cdot \sin(\mu \frac{\pi}{2}) = 0 \rightarrow \mu_n \cdot \frac{\pi}{2} = n\pi, \quad n = 1, 2, \ldots$

$\mu_n = 2n$

$$\begin{cases} \lambda_n = -(2n)^2 & , n = 1, 2, \ldots \quad \text{eigenvalues} \\ y_n(x) = \sin(2nx) & \quad \text{eigenfunctions} \end{cases}$$

Case 2: $\lambda = 0 \rightarrow y'' = 0 \rightarrow y = Ax + B$

$$\left.\begin{array}{l} y(0) = B = 0 \rightarrow B = 0 \\ y(\pi/2) = A \cdot \pi/2 = 0 \rightarrow A = 0 \end{array}\right\} \text{trivial Solution}$$

no eigenvalue/function for this case

Case 3: $\lambda < 0$ or $\lambda = -\mu^2 \rightarrow r = \pm\mu$

$y(x) = A\cosh(\mu x) + B\sinh(\mu x)$

$$\left.\begin{array}{l} y(0) = A \cdot 1 + B \cdot 0 = 0 \rightarrow A = 0 \\ y(\pi/2) = B \cdot \sinh(\mu \cdot \pi/2) = 0 \rightarrow B = 0 \end{array}\right\} \text{trivial Solution}$$

no eigenvalue/function

Question 3:

$$u_{tt} + k u_x = c^2 u_{xx}$$

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \frac{f^{(3)}(x)}{3!}\Delta x^3 + O(\Delta x^4) \quad (\mathrm{I})$$

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 - \frac{f^{(3)}(x)}{3!}\Delta x^3 + O(\Delta x^4) \quad (\mathrm{II})$$

(I) + (II) gives:

$$f(x+\Delta x) + f(x-\Delta x) = 2f(x) + f''(x)\Delta x^2 + O(\Delta x^4)$$

$$\Rightarrow f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} + O(\Delta x^2)$$

(I) − (II) gives,

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O(\Delta x^2)$$

Similarly:

$$f''(t) = \frac{f(x+\Delta t) - 2f(t) - f(x-\Delta t)}{\Delta t^2} + O(\Delta t^2)$$

Now, discretize the PDE:

$$u_{tt} + k u_x = c^2 u_{xx}$$

$$\frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{\Delta t^2} + O(\Delta t^2)$$

$$+k \frac{u(x+\Delta x,t) - u(x-\Delta x,t)}{2\Delta x} + O(\Delta x^2)$$

$$= c^2 \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} + O(\Delta x^2)$$

Now, find $u(x,t+\Delta t)$

$$u(x,t+\Delta t) = \left[-k\frac{\Delta t^2}{2\Delta x} + c^2\frac{\Delta t^2}{\Delta x^2}\right] u(x+\Delta x,t)$$

$$+ \left[-\frac{2c^2\Delta t^2}{\Delta x^2} + 2\right] u(x,t)$$

$$+ \left[\frac{c^2\Delta t^2}{\Delta x^2} + \frac{k\Delta t^2}{2\Delta x}\right] u(x-\Delta x,t)$$

$$- u(x,t-\Delta t) + O(\Delta t^2, \Delta x^2)$$

order of accuracy: $\begin{cases} \text{2nd order in space} \\ \text{2nd order in time} \end{cases}$

Alternative Solution:

Or we can choose a 1st order Scheme for $u_x$ and discretize the PDE as:

$$u_{tt} + k u_x = c^2 u_{xx}$$

$$\frac{u(x, t+\Delta t) - 2u(x,t) + u(x, t-\Delta t)}{\Delta t^2} + O(\Delta t^2)$$

$$+k \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x} + O(\Delta x)$$

$$= c^2 \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^2} + O(\Delta x^2)$$

Now, find $u(x, t+\Delta t)$

$$u(x, t+\Delta t) = \left[-k\frac{\Delta t^2}{2\Delta x} + c^2\frac{\Delta t^2}{\Delta x^2}\right] u(x+\Delta x, t)$$

$$+ \left[-\frac{2c^2\Delta t^2}{\Delta x^2} + 2 + \frac{k\Delta t^2}{\Delta x}\right] u(x,t)$$

$$+ \left[\frac{c^2\Delta t^2}{\Delta x^2}\right] u(x-\Delta x, t)$$

$$- u(x, t-\Delta t)$$

order of accuracy: $\begin{cases} \text{1st order in space} \\ \text{2nd order in time} \end{cases}$