

## Math 257/316, midterm 1, section 202

Februray 17, 2023 , Duration: 50 min , Total marks: 100 , Number of questions: 3

First Name:

Last Name:

SID:

Section:

Signature:

Problem	1	2	3	Total
Points				

### Instructions:

- Notes, calculators, phones, computers and your cheat sheets are not allowed.
- The formula sheet is on the last page of the exam booklet.
- Show all your work. A correct answer without the intermediate steps will not receive credit.

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. (50 marks) Consider the second order differential equation

$$Ly = 3x^2y'' + 2x(x+1)y' - 2y = 0, \quad (1)$$

(a) Classify the points  $0 \leq x < \infty$  as ordinary points, regular singular points and irregular singular points.

(b) Use the appropriate series expansion about the point  $x = 0$  to determine two linearly independent solutions to equation (1) . You only need to determine the first three non-zero terms in each case.

Question 1 (continued):

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2. (25 marks) Find all eigenvalues and corresponding eigenfunctions for the following eigenvalue problem

$$y'' + \lambda y = 0 \quad (0 < x < \pi/2), \quad y(0) = 0, \quad y(\pi/2) = 0. \quad (2)$$

Please show all the cases when solving the eigenvalue problem.

Question 2 (continued):

Question 2 (continued):

3. (25 marks) Consider the PDE for the wave equation with damping

$$u_{tt} + ku_x = c^2 u_{xx}, \quad (3)$$

- (a) Use the Taylor expansions of  $u$  to find the finite difference approximations for the derivatives in this equation and find a discrete form of the advection-diffusion equation.
- (b) Explain how you would advance the solution in time to find  $u(x, t + \Delta t)$  at each time step and discuss the order of accuracy of your numerical scheme in time and in space.

Taylor expansion:  $f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)\Delta x^2}{2!} + \frac{f'''(x)\Delta x^3}{3!} + O(\Delta x^4)$

Question 3 (continued):

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**Trigonometric and Hyperbolic Function identities**

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \sin \beta \cos \alpha$$

$$\sin^2 t + \cos^2 t = 1$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \beta \sin \alpha.$$

$$\sin^2 t = \frac{1}{2}(1 - \cos(2t))$$

$$\sinh(\alpha \pm \beta) = \sinh \alpha \cosh \beta \pm \sinh \beta \cosh \alpha$$

$$\cosh^2 t - \sinh^2 t = 1$$

$$\cosh(\alpha \pm \beta) = \cosh \alpha \cosh \beta \pm \sinh \beta \sinh \alpha.$$

$$\sinh^2 t = \frac{1}{2}(\cosh(2t) - 1)$$

**Basic linear ODE's with real coefficients**

	constant coefficients	Euler eq
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
indicial eq.	$ar^2 + br + c = 0$	$ar(r-1) + br + c = 0$
$r_1 \neq r_2$ real	$y = Ae^{r_1 x} + Be^{r_2 x}$	$y = Ax^{r_1} + Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx} + Bxe^{rx}$	$y = Ax^r + Bx^r \ln x $
$r = \lambda \pm i\mu$	$e^{\lambda x}[A \cos(\mu x) + B \sin(\mu x)]$	$x^\lambda[A \cos(\mu \ln x ) + B \sin(\mu \ln x )]$

Series solutions for  $y'' + p(x)y' + q(x)y = 0$  ( $\star$ ) around  $x = x_0$ .

Ordinary point  $x_0$ : Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

Regular singular point  $x_0$ : Rearrange ( $\star$ ) as:

$$(x - x_0)^2 y'' + [(x - x_0)p(x)](x - x_0)y' + [(x - x_0)^2 q(x)]y = 0$$

If  $r_1 > r_2$  are roots of the indicial equation:  $r(r-1) + br + c = 0$  where

$b = \lim_{x \rightarrow x_0} (x - x_0)p(x)$  and  $c = \lim_{x \rightarrow x_0} (x - x_0)^2 q(x)$  then a solution of ( $\star$ ) is

$$y_1(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r_1} \text{ where } a_0 = 1.$$

The second linearly independent solution  $y_2$  is of the form:

Case 1: If  $r_1 - r_2$  is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Case 2: If  $r_1 - r_2 = 0$ :

$$y_2(x) = y_1(x) \ln(x - x_0) + \sum_{n=1}^{\infty} b_n(x - x_0)^{n+r_2} \text{ for some } b_1, b_2, \dots$$

Case 3: If  $r_1 - r_2$  is a positive integer:

$$y_2(x) = a y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n(x - x_0)^{n+r_2} \text{ where } b_0 = 1.$$

**Fourier, sine and cosine series**

Let  $f(x)$  be defined in  $[-L, L]$  then its Fourier series  $Ff(x)$  is a  $2L$ -periodic function on  $\mathbf{R}$ :  $Ff(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\}$

where  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos(\frac{n\pi x}{L}) dx$  and  $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin(\frac{n\pi x}{L}) dx$

**Theorem (Pointwise convergence)** If  $f(x)$  and  $f'(x)$  are piecewise continuous, then  $Ff(x)$  converges for every  $x$  to  $\frac{1}{2}[f(x-) + f(x+)]$ .

**Parseval's identity**

$$\frac{1}{L} \int_{-L}^L |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

For  $f(x)$  defined in  $[0, L]$ , its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L}) dx,$$

$$Sf(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) dx.$$

**D'Alembert's solution to the wave equation**

**PDE:**  $u_{tt} = c^2 u_{xx}$ ,  $-\infty < x < \infty$ ,  $t > 0$  **IC:**  $u(x, 0) = f(x)$ ,  $u_t(x, 0) = g(x)$ .

**SOLUTION:**  $u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$

**Sturm-Liouville Eigenvalue Problems**

**ODE:**  $[p(x)y']' - q(x)y + \lambda r(x)y = 0$ ,  $a < x < b$ .

**BC:**  $\alpha_1 y(a) + \alpha_2 y'(a) = 0$ ,  $\beta_1 y(b) + \beta_2 y'(b) = 0$ .

**Hypothesis:**  $p$ ,  $p'$ ,  $q$ ,  $r$  continuous on  $[a, b]$ .  $p(x) > 0$  and  $r(x) > 0$  for  $x \in [a, b]$ .  $\alpha_1^2 + \alpha_2^2 > 0$ .  $\beta_1^2 + \beta_2^2 > 0$ .

**Properties** (1) The differential operator  $Ly = [p(x)y']' - q(x)y$  is symmetric in the sense that  $(f, Lg) = (Lf, g)$  for all  $f, g$  satisfying the BC, where  $(f, g) = \int_a^b f(x)g(x) dx$ . (2) All eigenvalues are real and can be ordered as  $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$  with  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ , and each eigenvalue admits a unique (up to a scalar factor) eigenfunction  $\phi_n$ .

(3) **Orthogonality:**  $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x) dx = 0$  if  $\lambda_m \neq \lambda_n$ .

(4) **Expansion:** If  $f(x) : [a, b] \rightarrow \mathbf{R}$  is square integrable, then

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \quad a < x < b, \quad c_n = \frac{\int_a^b f(x)\phi_n(x)r(x) dx}{\int_a^b \phi_n^2(x)r(x) dx}, \quad n = 1, 2, \dots$$