Problem 1:

$$u_t = u_{xx} + e^t \cos\left(\frac{5x}{2}\right) + 1 \qquad t>0, \quad 0 \leq x \leq \pi$$

$$u_x(0,t) = 0, \qquad u(\pi,t) = t$$

$$u(x,0) = 0$$

Remove the time-dependent B.C. :

$$w(x,t) = A(t)\,x + B(t)$$

$$w_x(0,t) = 0 \rightarrow A(t) = 0$$

$$w(\pi,t) = t \rightarrow B(t) = t$$

$$\rightarrow w(x,t) = t$$

Decompose the solution: $u(x,t) = w(x,t) + V(x,t)$

PDE : $u_t = w_t + V_t = (w_{xx} + V_{xx}) + e^t \cos\left(\frac{5x}{2}\right) + 1$

$$\cancel{1} + V_t = 0 + V_{xx} + e^t \cos\left(\frac{5x}{2}\right) \cancel{+1}$$

$$\longrightarrow V_t = V_{xx} + e^t \cos\left(\frac{5x}{2}\right)$$

B.C.'s : $u_x(0,t) = 0 = w_x(0,t) + V_x(0,t)$

$$0 = 0 + V_x(0,t)$$

$$\rightarrow V_x(0,t) = 0$$

$$u(\pi, t) = t = w(\pi, t) + v(\pi, t)$$

$$\cancel{t} = \cancel{t} + v(\pi, t)$$

$$\rightarrow \quad v(\pi, t) = 0$$

I.C.: $u(x, 0) = 0 = w(x, 0) + v(x, 0)$

$$0 = 0 + v(x, 0) \rightarrow v(x, 0) = 0$$

The B.V.P. for $v(x, t)$ is:

$$\begin{cases} v_t = v_{xx} + e^t \cos\left(\frac{5x}{2}\right) \\ v_x(0, t) = 0 = v(\pi, x) \\ v(x, 0) = 0 \end{cases}$$

eigenfunctions:

$$\rightarrow \cos\left(\frac{(2n+1)\pi}{2\pi} x\right) = \cos\left(\frac{2n+1}{2} x\right)$$

$$\mu_n = \frac{2n+1}{2}, \quad n = 0, 1, 2,$$

Expand the source term as eigenfunctions:

$$S(x, t) = e^t \cos\left(\frac{5x}{2}\right) = \sum_{n=0}^{\infty} S_n(t) \cos\left(\frac{2n+1}{2} x\right)$$

$$\rightarrow \quad S_n(t) = e^t \delta_{n2}$$

$$V(x,t) = \sum_{n=0}^{\infty} V_n(t) \cdot \cos\left(\frac{2n+1}{2} x\right)$$

substitute into PDE:

$$V_t - V_{xx} - S(x,t) = 0$$

$$\sum_{n=0}^{\infty} \left\{ V_n'(t) + \frac{(2n+1)^2}{4} V_n(t) - e^t \delta_{n2} \right\} \cos\left(\frac{2n+1}{2} x\right) =$$

Linear independency requires:

$$V_n'(t) + \frac{(2n+1)^2}{4} V_n(t) = e^t \delta_{n2}$$

Solve by the integrating factor method:

$$\left[V_n(t) \cdot e^{\frac{(2n+1)^2}{4} t} \right]' = e^t \cdot e^{\frac{(2n+1)^2}{4} t} \cdot \delta_{n2}$$

$$V_n(t) \cdot e^{\frac{(2n+1)^2}{4} t} = \delta_{n2} \int e^{\left(1 + \frac{(2n+1)^2}{4}\right) t} \, dt + C_n$$

$$V_n(t) = \frac{\delta_{n2}}{1 + \left(\frac{2n+1}{2}\right)^2} e^t + C_n e^{-\left(\frac{2n+1}{2}\right)^2 t}$$

or:

$$V(x,t) = \sum_{n=0}^{\infty} \left\{ \frac{\delta_{n2}}{1 + \left(\frac{2n+1}{2}\right)^2} \cdot e^t + C_n e^{-\left(\frac{2n+1}{2}\right)^2 t} \right\} \cos\left(\frac{2n+1}{2}\right.$$

Apply the I.C. :

$$v(x,0)=0 \quad \rightarrow \quad \frac{\delta_{n2}}{1+\left(\frac{2n+1}{2}\right)^2} + C_n = 0$$

$$\text{or} \quad C_n = -\frac{\delta_{n2}}{1+\left(\frac{2n+1}{2}\right)^2}$$

So, the solution to $v(x,t)$ problem is:

$$v(x,t) = \sum_{n=0}^{\infty} \left\{ \frac{\delta_{n2}}{1+\left(\frac{2n+1}{2}\right)^2} \left[e^{t} - e^{-\left(\frac{2n+1}{2}\right)^2 t} \right] \right\} \cos\left(\frac{2n+1}{2} x\right)$$

$$= \frac{1}{1+\frac{25}{4}} \left(e^{t} - e^{-\frac{25}{4}t} \right) \cos\left(\frac{5}{2} x\right)$$

$$u(x,t) = t + \frac{4}{29} \left(e^{t} - e^{-\frac{25}{4}t} \right) \cos\left(\frac{5}{2} x\right)$$

Problem 2:

$$u_{tt} = u_{xx}, \quad 0 \leq x \leq 2, \quad t > 0$$

B.C.'s: $u_x(0,t) = 0, \quad u_x(2,t) = 0$

I.C.'s: $u(x,0) = 0$

$$u_t(x,0) = g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$$

(a) $u(x,t) = X(x) \cdot T(t)$

$$\ddot{T} X = T X'' \longrightarrow \frac{\ddot{T}}{T} = \frac{X''}{X} = \lambda$$

X ODE: $\left.\begin{array}{l} X'' - \lambda X = 0 \\ X'(0) = X'(2) \end{array}\right\} \longrightarrow \begin{array}{l} X_n = \cos\left(\frac{n\pi x}{2}\right) \\ \lambda_n = -\left(\frac{n\pi}{2}\right)^2 \\ n = 1, 2, \ldots \end{array}$

$\lambda_0 = 0, \quad X_0 = 1$

T ODE: $\ddot{T} - \lambda T = 0$

$\lambda_0 = 0$: $\ddot{T} = 0 \longrightarrow T_0(t) = \frac{A_0}{2} t + \frac{B_0}{2}$

$\lambda_n = -\left(\frac{n\pi}{2}\right)^2 \quad \ddot{T} + \left(\frac{n\pi}{2}\right)^2 T = 0 \rightarrow T_n(t) = A_n \cos\left(\frac{n\pi}{2} t\right) + B_n \sin\left(\frac{n\pi}{2} t\right)$

So, the solution is:

$$u(x,t) = \frac{A_0}{2} t + \frac{B_0}{2} + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi}{2} t\right) + B_n \sin\left(\frac{n\pi}{2} t\right) \right\} \cos\left(\frac{n\pi x}{2}\right)$$

$$u_t(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ -A_n \frac{n\pi}{2} \sin\left(\frac{n\pi}{2} t\right) + B_n \frac{n\pi}{2} \cos\left(\frac{n\pi}{2} t\right) \right\} \cdot \cos\left(\frac{n\pi x}{2}\right)$$

Apply the I.C.'s:

$$u(x,0) = 0 = \frac{B_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{2}\right)$$

$$\longrightarrow \quad B_0 = 0, \quad A_n = 0$$

$$u_t(x,0) = f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left\{ B_n \frac{n\pi}{2} \right\}}_{C_n} \cos\left(\frac{n\pi x}{2}\right)$$

$$A_0 = \frac{2}{2}\int_0^2 g(x)\,dx = \int_1^2 1.\,dx = x\Big|_1^2 = 1$$

$$C_n = \frac{2}{2}\int_0^{\ell} g(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

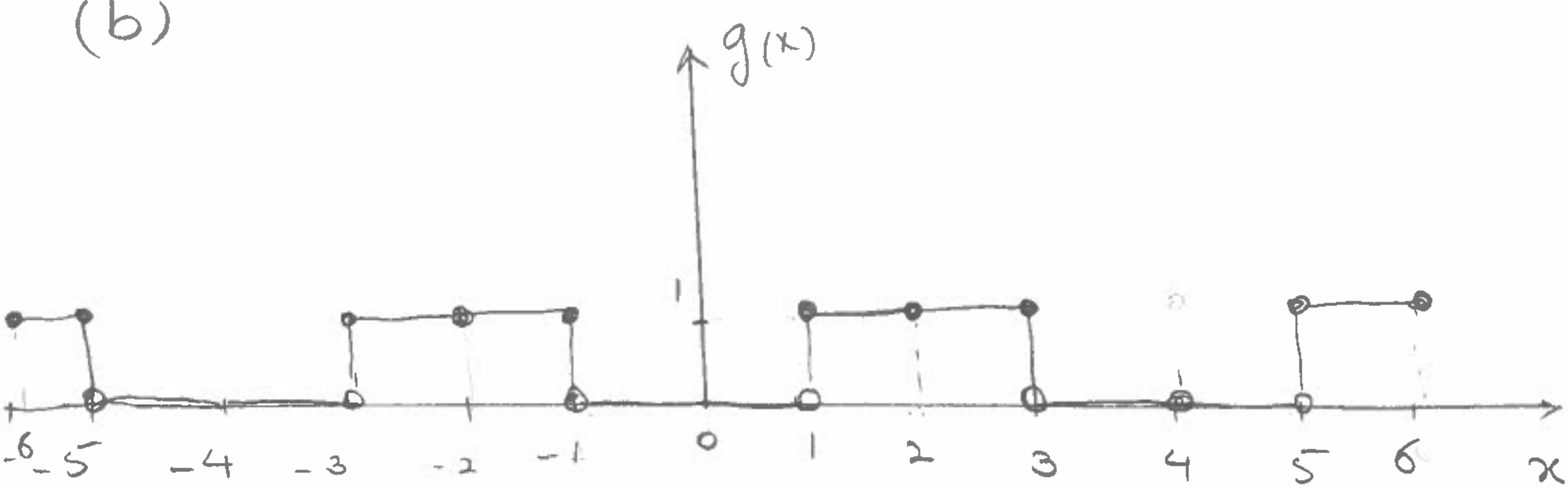
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2$$

$$= \frac{2}{n\pi}\left[\sin(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$B_n = \frac{C_n}{n\pi/2} = \frac{4}{(n\pi)^2}\left[0 - \sin\left(\frac{n\pi}{2}\right)\right]$$

$$u(x,t) = \frac{t}{2} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\left[0 - \sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi}{2}t\right)\cos\left(\frac{n\pi}{2}\right.$$

(b)



(c) $$u(x,t) = \frac{1}{2}\left\{ f^{e}(x-ct) + f^{e}(x+ct) \right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g^{e}(s)\,ds$$

$$u(1,4) = \frac{1}{2}\int_{1-4}^{1+4} g^{e}(s)\,ds = \frac{1}{2}\int_{-3}^{5} g^{e}(s)\,ds$$

$$u(1,4) = \frac{1}{2}\left\{\int_{-3}^{-1} 1.\,ds + \int_{1}^{3} 1.\,ds\right\} = \frac{1}{2}\left[s\Big|_{-3}^{-1} + s\Big|_{1}^{3} \right]$$
$$= \frac{1}{2}\left[2+2\right] = 2$$