Math 257/316, midterm 2, section 202

March 25, 2022 , Duration: 50 min , Total marks: 100 , Number of questions: 2

First Name: Last Name:

SID: Section: Signature:

Problem	1	2	Total
Points			

Instructions:

- Notes, calculators, phones, computers and your cheat sheets are not allowed.
- The formula sheet is on the last page of the exam booklet.
- Show all your work. A correct answer without the intermediate steps will not receive credit.

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. (60 marks) Consider the following inhomogeneous initial boundary value problem for the heat equation

$$
\begin{aligned}
u_t = u_{xx} + e^t \cos(\frac{5x}{2}) + 1 \qquad t > 0,\;\; 0 \le x \le \pi,\\
\text{BC:} \quad u_x(0,t) = 0, \;\; \text{and} \;\; u(\pi,t) = t\\
\text{IC:} \quad u(x,0) = 0
\end{aligned}
$$

Solve this problem by using an expansion in terms of the appropriate eigenfunctions, which correspond to the homogeneous form of the stated boundary conditions. (*Note:* Only show cases that give non-trivial solutions when solving the eigenvalue problems.)

Question 1 (continued):

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2. Consider the following initial boundary value problem for the wave equation

$$
\begin{aligned}
u_{tt} &= u_{xx} \qquad t > 0, \;\; 0 \le x \le \pi, \\
\text{BC:} \quad & u_x(0,t) = 0, \;\; \text{and} \;\; u_x(\pi,t) = 0 \\
\text{IC:} \quad & u(x,0) = \cos(x), \;\; \text{and} \;\; u_t(x,0) = 2\cos(x)
\end{aligned}
$$

(a) (30 marks) Solve this problem using the method of separation of variables. (*Note:* Only show cases that give non-trivial solutions when solving the eigenvalue problems.)
(b) (10 marks) Find the D'Alembert's solution (see the formula sheet) for the initial boundary value problem stated above. Compare this to the solution you found in (a).

Question 2 (continued):

Question 2 (continued):

Math 257-316 PDE Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\qquad \sin^2 t + \cos^2 t = 1$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\qquad \sin^2 t = \frac{1}{2}(1-\cos(2t))$
$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\qquad \cosh^2 t - \sinh^2 t = 1$
$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\qquad \sinh^2 t = \frac{1}{2}(\cosh(2t)-1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
indicial eq.	$ar^2 + br + c = 0$	$ar(r-1) + br + c = 0$
$r_1 \neq r_2$ real	$y = Ae^{r_1x} + Be^{r_2x}$	$y = Ax^{r_1} + Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx} + Bxe^{rx}$	$y = Ax^r + Bx^r\ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x) + B\sin(\mu x)]$	$x^\lambda[A\cos(\mu\ln\lvert x\rvert) + B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y'' + p(x)y' + q(x)y = 0$ $(\star)$ around $x = x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$$

Regular singular point x_0: Rearrange $(\star)$ as:
$(x-x_0)^2y'' + [(x-x_0)p(x)](x-x_0)y' + [(x-x_0)^2q(x)]y = 0$
If $r_1 > r_2$ are roots of the indicial equation: $r(r-1) + br + c = 0$ where $b = \lim_{x\to x_0}(x-x_0)p(x)$ and $c = \lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of $(\star)$ is

$$y_1(x) = \sum_{n=0}^\infty a_n(x-x_0)^{n+r_1} \quad \text{where } a_0 = 1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If $r_1 - r_2$ is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Case 2: If $r_1 - r_2 = 0$:

$$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^\infty b_n(x-x_0)^{n+r_2} \text{ for some } b_1, b_2...$$

Case 3: If $r_1 - r_2$ is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0) + \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L, L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$: $Ff(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}$
where $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin(\frac{n\pi x}{L})\,dx$

Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-) + f(x+)]$.

Parseval's indentity

$$\frac{1}{L}\int_{-L}^L |f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^\infty \left(|a_n|^2 + |b_n|^2\right).$$

For $f(x)$ defined in $[0, L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^\infty b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

D'Alembert's solution to the wave equation

PDE: $u_{tt} = c^2u_{xx},\ -\infty < x < \infty,\ t > 0$ **IC**: $u(x,0) = f(x),\ u_t(x,0) = g(x)$.
SOLUTION: $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

Sturm-Liouville Eigenvalue Problems

ODE: $[p(x)y']' - q(x)y + \lambda r(x)y = 0, \quad a < x < b.$
BC: $\alpha_1y(a) + \alpha_2y'(a) = 0, \quad \beta_1y(b) + \beta_2y'(b) = 0.$
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$. $\alpha_1^2 + \alpha_2^2 > 0$. $\beta_1^2 + \beta_2^2 > 0$.
Properties (1) The differential operator $Ly = [p(x)y']' - q(x)y$ is symmetric in the sense that $(f, Lg) = (Lf, g)$ for all f, g satisfying the BC, where $(f,g) = \int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = 0$ if $\lambda_m \neq \lambda_n$.
(4) **Expansion**: If $f(x) : [a,b] \to \mathbf{R}$ is square integrable, then

$$f(x) = \sum_{n=1}^\infty c_n\phi_n(x),\ a < x < b\ ,\ c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b \phi_n^2(x)r(x)\,dx},\ n = 1, 2, \ldots$$