Math 257/316 — Midterm Exam — 1 hr 15 min

June 03, 2025

- This test consists of 18 pages and 3 questions worth a total of 80 marks
- This is a closed-book examination. **Notes, calculators, phones, computers, or electronic device of any kind and cheat sheets are not allowed.**
- The formula sheet is on the last page of the exam booklet.

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1. $\boxed{\text{30 marks}}$ Consider the following Gauss's hypergeometric equation:

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0. \qquad (1)$$

where a, b and c are constants.

(a) Classify all points $\infty < x < \infty$ as ordinary, regular singular or irregular singular points. For the regular singular points, find the roots of the corresponding indicial equations. (6 marks)

Solution: Rewrite in standard form. Divide (1) by the leading coefficient $x(1-x)$ (for $x \neq 0, 1$):

$$y'' + P(x)\,y' + Q(x)\,y = 0,$$

$$P(x) = \frac{c-(a+b+1)\,x}{x(1-x)}, \quad Q(x) = -\frac{a\,b}{x(1-x)}.$$

The singular points are; $x = 0$, $x = 1$.

All other finite points $x \neq 0, 1$ are ordinary points.

- *Singular point at $x = 0$*

$$\lim_{x\mapsto 0} xP(x) = \lim_{x\mapsto 0} x\frac{c-(a+b+1)\,x}{x(1-x)} = c \quad \text{(finite)}$$

1mk (any limit computed)

$$\lim_{x\mapsto 0} x^2Q(x) = \lim_{x\mapsto 0} -x^2\,\frac{a\,b}{x(1-x)} = 0 \quad \text{(finite)}.$$

$x = 0$ is a **regular singular point**. 1mk (depend on limit)

Indicial equation

$$r(r-1) + cr = 0$$
$$\Longrightarrow r_1 = 0 \quad \text{and} \quad r_2 = 1 - c$$

1mk

- *Singular point at $x = 1$.*

$$\lim_{x\mapsto 1}(x-1)\frac{c-(a+b+1)\,x}{x(1-x)} = a+b+1-c \quad \text{(finite)}$$

1mk

$$\lim_{x\mapsto 1} -(x-1)^2\,\frac{a\,b}{x(1-x)} = 0 \quad \text{(finite)}.$$

$x = 1$ is a **regular singular point**. 1mk

Indicial equation

$$r(r-1) + (a+b+1-c)r = 0$$
$$\Longrightarrow r_1 = 0 \quad \text{and} \quad r_2 = c - a - b$$

1mk

(b) Classify the points at infinity as ordinary, regular singular or irregular singular points. If they are regular singular points, find the roots of the indicial equation. (6 marks)

Solution: Let $t = 1/x$, so $x = 1/t$ and $y(x) = Y(t)$. Using

$$\frac{dy}{dx} = -\,t^2\,\frac{dY}{dt}, \qquad \frac{d^2y}{dx^2} = 2\,t^3\,\frac{dY}{dt} + t^4\,\frac{d^2Y}{dt^2},$$

The differential equation becomes

$$(t-1)\,t^2\,Y'' \;+\; \big[2\,t\,(t-1) - c\,t^2 + (a+b+1)\,t\big]\,Y' \;-\; a\,b\,Y \;=\; 0.$$

3mks (all correct) [cut off at page edge]

Divide by $(t-1)\,t^2$ to write in standard form:

Any attempt – 2mks

$$Y'' + P(t)\,Y' + Q(t)\,Y = 0,$$

$$P(t) = \frac{(2-c)\,t + (a+b-1)}{t\,(t-1)}, \quad Q(t) = -\,\frac{a\,b}{t^2\,(t-1)}.$$

Classification at $t = 0$ (i.e. $x = \infty$).

Compute

$$\lim_{t\to 0} \frac{(2-c)\,t + (a+b-1)}{t-1} = 1 - (a+b) \quad \text{(finite)},$$

1mk

$$\lim_{t\to 0}\Big[-\,\frac{a\,b}{t-1}\Big] = a\,b \quad \text{(finite)}.$$

Hence $t = 0$ (i.e. $x = \infty$) is a **regular singular point**.

1mk (depend on limits above)

Indicial equation at $t = 0$.

$$r(r-1) + (1-(a+b))r + ab = 0 \implies (r-a)(r-b) = 0$$
$$\implies r_1 = a \quad \text{and} \quad r_2 = b$$

1mk (mark according to limits)

(c) Given that $a = 1$, $b = 1$ and $c = 1$. Use appropriate series expansion to determine a series solution to (1) that satisfies $y(0) = 0.5$. What is the radius of convergence of this series? (**You may choose to write the series in the general form, or only determine the first three non-zero terms in each case.**) (18 marks)

Hints: The following hints may be useful: Given $x = 1/t$,

$$\frac{dy}{dx} = -t^2\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = 2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}$$

Solution: With $a = 1$, $b = 1$, and $c = 1$, Equation (1) becomes

$$x(1-x)\,y'' + \big(1-3x\big)\,y' - y \;=\; 0.$$ 1mk

Let

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+r},\;\; y'(x) = \sum_{n=1}^{\infty} a_n(n+r)x^{\,n+r-1},$$

$$y''(x) = \sum_{n=2}^{\infty} a_n(n+r)(n+r-1)\,x^{n+r-2}.$$ 3mks

Substitute into the ODE:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} - \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r}$$

$$+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - 3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

Simplify

$$\sum_{n=0}^{\infty}(n+r)(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}[(n+r)(n+r+2)+1]a_n x^{n+r} = 0$$ 2mks

Shift index of the first summation, let $n+r-1 = m+r$ and simplify 1mk

$$a_0 r^2 x^{r-1}$$

$$+\sum_{n=0}^{\infty}\{a_{n+1}(n+r+1)^2 - a_n[(n+r)(n+r+2)+1]\}x^{n+r} = 0$$ 1mk

Since difference x powers are independent, we equate x coefficients to zero

x^{r-1}**: Indicial equation:**
$a_0 r^2 = 0 \implies r_{1,2} = 0$ 1mk

x^{n+r}**: Recurrence relation**

$$a_{n+1} = \frac{a_n((n+r)(n+r+2)+1)}{(n+r+1)^2}, \quad n \geq 0$$ 1mk

Note that we are in Case: $r_1 - r_2 = 0$
Case 1: $r_1 = 0$**:** The recurrence relation simplifies to

$$a_{n+1} = \frac{a_n(n+1)^2}{(n+1)^2} = a_n, \quad n \geq 0$$ 1mk

Hence,

$$a_n = a_0, \quad \forall\, n \geq 0.$$

Therefore

$$y_1(x) = \sum_{n=0}^{\infty} a_0\, x^n = \frac{a_0}{1-x}, \quad |x| < 1.$$ 1mk

Case 2: $r_1 = 0$**:** Repeated root

$$y_2(x) = \ln x \cdot y_1(x) + \sum_{n=0}^{\infty} b_n x^n$$ 2mks

The general solution is

$$\boxed{y(x) = c_1 y_1(x) + c_2 y_2(x)}$$ 1mk

Since $y(0) = 0.5$ is finite, set $c_2 = 0$ 1mk
We get

$$y(x) = c_1 y_1(x)$$

Th initial condition $y(0) = 0.5$ gives $c_1 = 0.5$. Thus

$$\boxed{y(x) \;=\; 0.5 \sum_{n=0}^{\infty} x^n \;=\; \frac{0.5}{1-x}}$$ 1mk

The series $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$. Hence the radius of convergence is

$$\boxed{R = 1.}$$ 1mk

2. 30 marks Apply the method of separation of variables to determine the solution to the one dimensional heat equation with the following Mixed homogeneous boundary conditions (Show all cases of the eigenvalue problem):

$$\text{P.D.E.:} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi,\ t > 0 \tag{2a}$$

$$\text{B.C. :} \quad \frac{\partial u(0,t)}{\partial x} = 0 = u(\pi, t) \tag{2b}$$

$$\text{I.C.:} \quad u(x,0) = x(\pi - x) \tag{2c}$$

Hint: It may be useful to know that:

$$\frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos\left(\frac{2n+1}{2}x\right) dx = \frac{8}{\pi}\frac{4(-1)^n - (2n+1)\pi}{(2n+1)^3}$$

Solution:

Use separation of variables: Let

$$u(x,t) \;=\; X(x)\,T(t).$$

Substituting into $u_t = u_{xx}$ gives

$$X(x)\,T'(t) \;=\; X''(x)\,T(t) \quad\Longrightarrow\quad \frac{T'(t)}{T(t)} \;=\; \frac{X''(x)}{X(x)} \;=\; -\lambda^2,$$

where λ^2 is a separation constant. Then:

$$T'(t) + \lambda^2\,T(t) = 0 \implies T(t) = D\,e^{-\lambda^2 t}.$$

The eigenvalue problem:

$$X''(x) + \lambda^2\,X(x) = 0, \quad X'(0) = 0,\ X(\pi) = 0.$$

The general solution of $X'' + \lambda^2 X = 0$ is

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x), \quad X'(x) = -A\,\lambda\sin(\lambda x) + B\,\lambda\cos(\lambda x).$$

Impose $X'(0) = 0$:

$$X'(0) = -A\,\lambda\sin(0) + B\,\lambda\cos(0) = B\,\lambda = 0 \implies B = 0.$$

Hence $X(x) = A\cos(\lambda x)$. Next impose $X(\pi) = 0$:

$$X(\pi) = A\cos(\lambda\pi) = 0 \implies \cos(\lambda\pi) = 0 \implies \lambda\pi = \frac{\pi}{2} + k\pi,\ k = 0, 1, 2, \ldots$$

Thus

$$\lambda_k = \frac{2k+1}{2}, \quad k = 0, 1, 2, \ldots$$

and a corresponding (nontrivial) eigenfunction is

$$X_k(x) \;=\; \cos\!\Big(\tfrac{2k+1}{2}\,x\Big), \quad k = 0, 1, 2, \ldots$$

The time-dependent factor is then

$$T_k(t) = e^{-\lambda_k^2\,t} = e^{-\left(\frac{2k+1}{2}\right)^2 t}.$$

Hence the general solution is

$$u(x,t) = \sum_{k=0}^{\infty} A_k \,\cos\!\Big(\tfrac{2k+1}{2}\,x\Big)\; e^{-\left(\frac{2k+1}{2}\right)^2 t}.$$

To satisfy the initial condition $u(x,0) = x(\pi - x)$, we require

$$x(\pi - x) = \sum_{k=0}^{\infty} A_k \,\cos\!\Big(\tfrac{2k+1}{2}\,x\Big), \quad 0 \le x \le \pi.$$

Since the eigenfunctions $\{\cos((2k+1)x/2)\}$ are orthogonal on $[0,\pi]$ with weight 1, the coefficients A_k are given by

$$A_k = \frac{\displaystyle\int_0^{\pi} x(\pi - x)\,\cos\!\Big(\tfrac{2k+1}{2}\,x\Big)\,dx}{\displaystyle\int_0^{\pi} \cos^2\!\Big(\tfrac{2k+1}{2}\,x\Big)\,dx}.$$

Compute the denominator first:

$$\int_0^{\pi} \cos^2\!\Big(\tfrac{2k+1}{2}\,x\Big)\,dx = \frac{\pi}{2}.$$

Thus

$$A_k = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,\cos\!\Big(\tfrac{2k+1}{2}\,x\Big)\,dx.$$

An integration by parts yields

$$\int_0^{\pi} x(\pi - x)\,\cos\!\Big(\tfrac{2k+1}{2}\,x\Big)\,dx = \frac{4(-1)^k - (2k+1)\,\pi}{(2k+1)^3}\cdot 4.$$

Hence

$$A_k = \frac{2}{\pi}\cdot\frac{4\big[\,4(-1)^k - (2k+1)\,\pi\big]}{(2k+1)^3} = \frac{8}{\pi}\,\frac{4(-1)^k - (2k+1)\,\pi}{(2k+1)^3}.$$

The final solution is therefore given by

$$\boxed{u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty}\left[\frac{4(-1)^k - (2k+1)\,\pi}{(2k+1)^3}\right]\cos\!\Big(\tfrac{2k+1}{2}\,x\Big)\;\exp\!\Big[-\big(\tfrac{2k+1}{2}\big)^2\,t\Big].}$$

3. 20 marks Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Using the Taylor's expansion, prove that

$$f'(x) = \frac{2f(x+3h) - 9f(x+2h) + 18f(x+h) - 11f(x)}{6h} + O\left(h^p\right),$$

and the deduce the order of accuracy p.

Hint: Taylor expansion: It may be useful to know that

$$f(x+kh) = f(x)+kh\, f'(x)+\frac{(kh)^2}{2}f''(x)+\frac{(kh)^3}{6}f'''(x)+\frac{(kh)^4}{24}f^{(4)}(x)+\cdots;\ k = 1, 2, 3\cdots$$

Solution: Expand $f(x+h)$, $f(x+2h)$, and $f(x+3h)$ about x via Taylor's theorem:

$$f(x+h) = f(x) + h\, f'(x) + \tfrac{h^2}{2} f''(x) + \tfrac{h^3}{6} f^{(3)}(x) + \tfrac{h^4}{24} f^{(4)}(x) + O(h^5),$$
$$f(x+2h) = f(x) + 2h\, f'(x) + 2h^2 f''(x) + \tfrac{4h^3}{3} f^{(3)}(x) + \tfrac{2h^4}{3} f^{(4)}(x) + O(h^5),$$
$$f(x+3h) = f(x) + 3h\, f'(x) + \tfrac{9h^2}{2} f''(x) + \tfrac{9h^3}{2} f^{(3)}(x) + \tfrac{27h^4}{8} f^{(4)}(x) + O(h^5).$$

Substitute the Taylor's expansion into

$$N = 2\,f(x+3h) - 9\,f(x+2h) + 18\,f(x+h) - 11\,f(x).$$

Substitute the expansions:

$$2\,f(x+3h) = 2f(x) + 6h\, f'(x) + 9h^2 f''(x) + 9h^3 f^{(3)}(x) + \tfrac{27h^4}{4} f^{(4)}(x) + O(h^5),$$
$$-9\,f(x+2h) = -9f(x) - 18h\, f'(x) - 18h^2 f''(x) - 12h^3 f^{(3)}(x) - 6h^4 f^{(4)}(x) + O(h^5),$$
$$18\,f(x+h) = 18f(x) + 18h\, f'(x) + 9h^2 f''(x) + 3h^3 f^{(3)}(x) + \tfrac{3h^4}{4} f^{(4)}(x) + O(h^5),$$
$$-11\,f(x) = -11f(x).$$

The term-by-term summation gives:

Coefficient of $f(x)$:

$$2 - 9 + 18 - 11 = 0.$$

Coefficient of $h\,f'(x)$:

$$6 - 18 + 18 = 6.$$

Coefficient of $h^2\, f''(x)$:

$$9 - 18 + 9 = 0.$$

Coefficient of $h^3 f^{(3)}(x)$:

$$9 - 12 + 3 = 0.$$

Coefficient of $h^4 f^{(4)}(x)$:

$$\tfrac{27}{4} - 6 + \tfrac{3}{4} = \tfrac{27+3}{4} - 6 = \tfrac{30}{4} - 6 = \tfrac{30-24}{4} = \tfrac{6}{4} = \tfrac{3}{2}.$$

Thus

$$N = 6h\, f'(x) + \frac{3}{2} h^4 f^{(4)}(x) + O(h^5).$$

Divide by $6h$:

$$\frac{2\, f(x+3h) - 9\, f(x+2h) + 18\, f(x+h) - 11\, f(x)}{6h} = f'(x) + \frac{3}{2} \frac{h^4}{6h} f^{(4)}(x) + O(h^4)$$

$$= f'(x) + \frac{1}{4} h^3 f^{(4)}(x) + O(h^4).$$

Hence

$$f'(x) = \frac{2\, f(x+3h) - 9\, f(x+2h) + 18\, f(x+h) - 11\, f(x)}{6h} + O(h^3).$$

The finite difference approximation has order of accuracy

$$\boxed{p = 3.}$$

Math 257-316 PDE   Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\sin^2 t + \cos^2 t = 1$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\sin^2 t = \frac{1}{2}(1 - \cos(2t))$

$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\cosh^2 t - \sinh^2 t = 1$

$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\sinh^2 t = \frac{1}{2}(\cosh(2t) - 1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
indicial eq.	$ar^2 + br + c = 0$	$ar(r-1) + br + c = 0$
$r_1 \neq r_2$ real	$y = Ae^{r_1 x} + Be^{r_2 x}$	$y = Ax^{r_1} + Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx} + Bxe^{rx}$	$y = Ax^r + Bx^r \ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x) + B\sin(\mu x)]$	$x^\lambda[A\cos(\mu\ln\lvert x\rvert) + B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y'' + p(x)y' + q(x)y = 0$ $(\star)$ around $x = x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$$

Regular singular point x_0: Rearrange $(\star)$ as:
$(x-x_0)^2y'' + [(x-x_0)p(x)](x-x_0)y' + [(x-x_0)^2q(x)]y = 0$
If $r_1 > r_2$ are roots of the indicial equation: $r(r-1) + br + c = 0$ where $b = \lim_{x\to x_0}(x-x_0)p(x)$ and $c = \lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of $(\star)$ is

$$y_1(x) = \sum_{n=0}^\infty a_n(x-x_0)^{n+r_1} \quad \text{where } a_0 = 1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If $r_1 - r_2$ is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Case 2: If $r_1 - r_2 = 0$:

$$y_2(x) = y_1(x)\ln(x-x_0) + \sum_{n=1}^\infty b_n(x-x_0)^{n+r_2} \text{ for some } b_1, b_{2\ldots}$$

Case 3: If $r_1 - r_2$ is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0) + \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } b_0 = 1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L, L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$: $Ff(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}$
where $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin(\frac{n\pi x}{L})\,dx$

Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-) + f(x+)]$.

Parseval's indentity

$$\frac{1}{L}\int_{-L}^L |f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^\infty\left(|a_n|^2 + |b_n|^2\right).$$

For $f(x)$ defined in $[0, L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^\infty b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

D'Alembert's solution to the wave equation

PDE: $u_{tt} = c^2u_{xx}$, $-\infty < x < \infty$, $t > 0$ **IC**: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.
SOLUTION: $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

Sturm-Liouville Eigenvalue Problems

ODE: $[p(x)y']' - q(x)y + \lambda r(x)y = 0$, $a < x < b$.
BC: $\alpha_1 y(a) + \alpha_2 y'(a) = 0$, $\beta_1 y(b) + \beta_2 y'(b) = 0$.
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x) > 0$ and $r(x) > 0$ for $x \in [a,b]$. $\alpha_1^2 + \alpha_2^2 > 0$. $\beta_1^2 + \beta_2^2 > 0$.
Properties (1) The differential operator $Ly = [p(x)y']' - q(x)y$ is symmetric in the sense that $(f, Lg) = (Lf, g)$ for all f, g satisfying the BC, where $(f,g) = \int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m, r\phi_n) = \int_a^b \phi_m(x)\phi_n(x)r(x)\,dx = 0$ if $\lambda_m \neq \lambda_n$.
(4) **Expansion**: If $f(x) : [a,b] \to \mathbf{R}$ is square integrable, then

$$f(x) = \sum_{n=1}^\infty c_n\phi_n(x), \ a < x < b, \ c_n = \frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b \phi_n^2(x)r(x)\,dx}, \ n = 1, 2, \ldots$$