# Math 257/316 Assignment 1, 2025

## Due Tuesday May 20, 2025. Submit online in a PDF document on Canvas by 11:59 pm of the due date

**Problem 1 (Do not Submit):** (**ODE Review**) Find the general solutions of the following equations:

a. $\sqrt{1-x^2}\, y' - xy = 0$ for $x \in (0,1)$.
b. $y'' + 2y' + y = e^{-2x}$.
c. $2x^2y'' + 4xy' - y = x$.

**Problem 2 (Submit):** (**Power Series Solution**): Consider

$$(x^2+1)y' + y = 0$$

for $x \in (-1,1)$.

a. Use the methods you learned from ODE Review to find a general solution to the equation.

b. Find the first three coefficients of the Taylor series expansion of this solution. You may either calculate by hand or use an online tool (Taylor series calculator at wolframalpha.com).

c. Now assume a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Obtain a recursion formula for the coefficients $a_n$. Compute the first three coefficients using the recursion formula.

**Problem 3 (Submit):**(**Power Series Solution**) Consider

$$y' + \frac{1}{x}y = 0.$$

a. Find the general solution to the equation using techniques from ODE Review.

b. Use power series expansion to find a solution. Is it the same solution you obtained in part a. ?

**Problem 4 (Do not submit):** (**Power Series Solution**): Consider the following first order linear ODEs:

$$\begin{aligned} y' + (1-2x)y &= 0 \quad (1) \\ xy' + (2-x)y &= 0 \quad (2) \end{aligned}$$

a. Solve the differential equations (1) and (2) using the appropriate integrating factors.

b. Expand the solution to (1) as Taylor series about the point $x_0 = 0$. Expand the exponential in the solution to (2) as a power series.

c. Now for (1) assume a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (3)$$

obtain a recursion for the coefficients $a_n$. Use these recursions to determine the series representation of the solution. Compare this result to the series obtained in part b above.

d. Try using the same power series expansion (3) to solve (2). What happens?

e. Consider the following recursive strategy to generate an approximate solution to (2). Rewrite (2) as

$$xy' + 2y = xy \quad (4)$$

Now assuming $x \to 0$ and discarding the right hand side of (4), find a first order approximation $y_0$ as the solution to

$$xy_0' + 2y_0 = 0$$

Now substitute $y_0$ on the right side of (4) and solve for $y_1$

$$xy_1' + 2y_1 = xy_0$$

Continue this process till you obtain $y_2$. How does $y_2$ compare with the series solution to (2) obtained in b? Can you use this series to motivate a modification to the series expansion (3) that would be appropriate to use to obtain a series solution to (2)?

# Math 257/316 Assignment 1

Grading Scheme

September 17, 2024

- Problem 1 (3 points)

  (1 point) (a): $y = Ce^{-\sqrt{1-x^2}}$.

  (1 point) (b): $y = C_1 e^{-x} + C_2 x e^{-x} + y_p$. Any correct $y_p$ suffices (e.g. $e^{-2x}$) (partial credit: correct homogeneous/particular solution)

  (1 point) (c): $y = x^{-\frac{1}{2}}(C_1 x^{\frac{\sqrt{3}}{2}} + C_2 x^{-\frac{\sqrt{3}}{2}}) + y_p$. (e.g. $y_p = \frac{x}{3}$) (partial credit: correct homogeneous/particular solution)

- Problem 2 (4 points)

  (1 point) (a): $y = \frac{C}{e^{arctanx}}$.

  (1 point) (b): $a_0 = c, a_1 = -c, a_2 = \frac{c}{2}$. (full credit: $a_0$ is not taken as a constant; partial credit: not all coefficients are wrong)

  (1 point) (c) recursion formula: $a_{n+1} = \frac{-(n-1)a_{n-1} - a_n}{n+1}$.

  (1 point) coefficients (c): $a_1 = -a_0$, $a_2 = \frac{a_0}{2}$. (full credit: $a_0$ is not taken as a constant; partial credit: not all coefficients are wrong)

# A Assignment 1 Solution

**Problem 1:**

(a) The system is separable. Integrate

$$\frac{1}{y}y' = \frac{x}{\sqrt{1-x^2}}$$

and obtain

$$y = Ce^{-\sqrt{1-x^2}}.$$

(b) Use the ansatz $y = e^{rx}$ to deduce from the characteristic equation $r = -1$ (repeated root). Hence, the general solution has the form

$$y_h = C_1 e^{-x} + C_2 x e^{-x}.$$

Note $e^{-2x}$ satisfies the equation, and hence it is a particular solution. Therefore,

$$y = C_1 e^{-x} + C_2 x e^{-x} + e^{-2x}.$$

(c) Use the ansatz $y = x^r$ and deduce (by substituting into the homogeneous equation) $r = \frac{-1\pm\sqrt{3}}{2}$. Hence, the general solution is

$$y_h = C_1 x^{\frac{-1+\sqrt{3}}{2}} + C_2 x^{\frac{-1-\sqrt{3}}{2}}.$$

Note $\frac{x}{3}$ satisfies the equation, and so it is a particular solution. Therefore,

$$y = C_1 x^{\frac{-1+\sqrt{3}}{2}} + C_2 x^{\frac{-1-\sqrt{3}}{2}} + \frac{x}{3}.$$

**Problem 2:**

(a) Compute the integrating factor

$$\mu(x) = e^{\int \frac{1}{x^2}\,dx} = e^{arctan(x)}.$$

Multiplying the factor with both sides of the equation, we have

$$(e^{arctan(x)} y)' = 0.$$

Therefore,

$$y = \frac{C}{e^{arctan(x)}}.$$

(b) Assume the expansion is centered around $x = 0$. Then, $a_0 = c, a_1 = -c, a_2 = \frac{c}{2}$.

(c) Assume a power series expansion around 0 and substitute into the equation. We have

$$(x^2+1)\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Multiplying $(x^2+1)$ into the sum and rearranging, we have

$$\sum_{n=2}^{\infty}((n-1)a_{n-1} + a_n + (n+1)a_{n+1})x^n + a_0 + a_1 + (2a_2 + a_1)x = 0.$$

By the linear independence of $\{x^n\}_n$, this implies

$$a_1 = -a_0, a_2 = \frac{a_0}{2},$$

$$a_{n+1} = \frac{-(n-1)a_{n-1} - a_n}{n+1}.$$

The coefficients match with the result in (b).

**Problem 3:**

(a) The system is separable. By integration, we have

$$y = \frac{C}{|x|}.$$

(b) Assume a power series expansion around 0 and substitute into the equation. Upon simplification, we have

$$\frac{1}{x}a_0 + \sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = 0.$$

Again by linear independence, we deduce

$$a_n = 0 \quad \forall n.$$

This means the power series solution is the trivial solution, which is not the solution we obtained in (a).

## Problem 3:

**a.** (1): $y' + (1-2x)y = 0$

the integrating factor: $\mu(x) = e^{\int (1-2x)dx}$

$= e^{x-x^2}$

$e^{x-x^2} y' + e^{x-x^2} \cdot (1-2x)y = 0$

$(e^{x-x^2} \cdot y)' = 0 \rightarrow e^{x-x^2} y = c$

or $y(x) = c \cdot e^{x^2-x}$

(2): $xy' + (2-x)y = 0$ , $y' + \frac{2-x}{x} y = 0$

the integrating factor: $\mu(x) = e^{\int (\frac{2}{x} - 1)dx}$

$= e^{2\ln|x| - x}$

$= x^2 \cdot e^{-x}$

$x^2 \cdot e^{-x} y' + x e^{-x} (2-x) y = 0$

$(x^2 e^{-x} \cdot y)' = 0 \rightarrow y(x) = c\, x^{-2} \cdot e^{x}$

**b.** (1) $y(x) = c\left[1 - x + \frac{3}{2}x^2 + \cdots\right]$

(2) $y(x) = c\, x^{-2}\left[1 + x + \frac{x^2}{2} + \cdots\right]$

c. $y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$\mathcal{L}y = y' + y - 2xy = 0$

$$\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$m = n-1$
$n = m+1$
$n=1 \to m=0$

$n = m$

$m = n+1$
$n = m-1$
$n=0 \to m=1$

$$\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^m - 2\sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$a_1 x^0 + a_0 \cdot x^0 + \sum_{m=1}^{\infty}\left[a_{m+1}(m+1) + a_m - 2a_{m-1}\right]x^m = 0$$

$x^0]\quad a_1 = -a_0$

$x^m]\quad a_{m+1} = \dfrac{2a_{m-1} - a_m}{m+1} \qquad m \geq 1$

$m=1: \quad a_2 = \dfrac{2a_0 - a_1}{2} = \dfrac{3a_0}{2}$

$m=2: \quad a_3 = \dfrac{2a_1 - a_2}{2} = \dfrac{-2a_0 - \frac{3}{2}a_0}{2} = -\dfrac{7}{6}a_0$

$\vdots$

$y(x) = a_0\left[1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \cdots\right]$

The same as what we found in (b).

(d) $Ly = xy' + 2y - xy = 0$

$$\sum_{n=1}^{\infty} a_n n x^n + 2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$m = n$ $\quad$ $m=n$ $\quad$ $m = n+1$, $n = m-1$, $n=0 \rightarrow m=1$

$$\sum_{m=1}^{\infty} a_m \cdot m x^m + 2\sum_{m=0}^{\infty} a_m x^m - \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$2a_0 x^0 + \sum_{m=1}^{\infty} [a_m \cdot m + 2a_m - a_{m-1}] x^m = 0$$

$x^0]$ $\quad a_0 = 0$

$x^m]_{m\geq 1}$ $\quad a_m = \dfrac{a_{m-1}}{(m+2)}$

$m=1 \rightarrow a_1 = \dfrac{a_0}{3} = 0$

$m=2 \rightarrow a_2 = \dfrac{a_1}{4} = 0$, ...

So only gives a trivial solution: $y(x) = 0$ ...

e) $xy' + 2y = xy$

$Ly_0 = xy_0' + 2y_0 = 0 \rightarrow \dfrac{dy_0}{dx} = -\dfrac{2y_0}{x}$ $\quad$ separable

$\dfrac{dy_0}{y_0} = -\dfrac{2dx}{x} \rightarrow \ln|y_0| = -2\ln|x| + c \rightarrow y_0 = C_0 x^{-2}$

$Ly_1 = xy_1' + 2y_1 = xy_0 = C_0 x^{-1}$

the integrating factor: $\mu(x) = e^{\int \frac{2}{x} \cdot dx} = e^{2\ln|x|} = x^2$

$$(x^2 y_1)' = c_0 x^{-2} \cdot x^2 = c_0 \rightarrow x^2 y_1 = c_0 x + c_1$$

$$\text{or } y_1 = x^{-2}[c_0 x + c_1]$$

$$= c_0 x^{-2}[1 + x]$$

[Note: $c_1 = c_0$ since there is only one solution]

$$L y_2 = x y_2' + 2 y_2 = x y_1 = c_0 x^{-1}[1+x]$$

we can use the same integrating factor here:

$$(x^2 y_2)' = c_0 (1+x) \rightarrow x^2 y_0 = c_0\left(x + \frac{x^2}{2}\right) + c_2$$

again: $c_2 = c_0$

$$y_2 = c_0 x^{-2}\left(1 + x + \frac{x^2}{2}\right)$$

The first 3 terms are the same as the ones we found in (b).

This gives us the idea that we should use a series solution of the form: $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ (Frobenius series).