Math 257/316 Assignment 2, 2025

Due Tuesday May 27 Submit online in a PDF document on Canvas by 11:59 pm of the due date

Problem 1 (Do not submit)– Power series: Consider the ODE:

$$(1+x^3)y'' - 6xy = 0. \tag{1}$$

(a) Compute the first 3 nonzero terms of power series expansion about $x = 0$ for two linearly independent solutions.
(b) Use the ratio test to determine the radius of convergence of the series. Could your result have been predicted by inspection?

Problem 2 (Submit)– Power series: Consider the Differential equation

$$(1+x^2)y'' + xy' - y = 0. \tag{2}$$

(a) Find the first 3 nonzero terms of the power series expansion of the general solution about $x = 0$.
(b) Use the ratio test to determine the radius of convergence of the series. What can you say about the radius of convergence without solving the ODE?
(c) Determine the solution that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 0$.

Problem 3 (Do not submit)– Power series: Compute the first 3 nonzero terms of the power series expansion about $x = 0$ of two linearly independent solutions of the ODE: $y'' - (\sin x)\, y = 0$.

Problem 4 (Submit)– Frobenius series: Consider the following Gauss's hypergeometric equation:

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0. \tag{3}$$

(a) Classify the points $0 < x < \infty$ as ordinary points, regular singular points, or irregularsingular points.
(b) Find the exponents at the regular singular point(s)
(c) Assuming that $1 - c$ is not a positive integer, find the series solution of (3), in the neighborhood of $x = 0$. What would you expect the radius of convergence of this series to be?
(d) (**DO NOT SUBMIT**) Assuming that $1 - c$ is not an integer or zero, find the second solution for $0 < x < 1$.

Problem 5 (Do not submit)–Singular Points at Infinity: (a) Consider the equation

$$y'' + \frac{4}{x}y' + \frac{2}{x^2}y = 0.$$

Use the change of variable $t = \frac{1}{x}$ to identify and classify singular points at $x = \infty$. Compute the indices of the associated Frobenius series (i.e. if $y = x^r \sum a_n x^n$, determine r).

(b) Verify that

$$y'' + \frac{c-x}{x}y' - \frac{a}{x}y = 0$$

has an irregular singular point at $x = \infty$.

Problem 6 (Do not submit)–Bessel of one-half: Consider the Bessel equation of order 1/2

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu = \frac{1}{2}, \quad x > 0 \tag{4}$$

(a) Show that (4) can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this, conclude that

$$y_1(x) = x^{-1/2}\cos x \quad \text{and} \quad y_2(x) = x^{-1/2}\sin x$$

are solutions of the Bessel equation of order one-half.
(b) Show that $x = 0$ is a regular singular point. Find the roots of the indicial equation for $x = 0$.
(c) Use a Frobenius series expansion about $x = 0$ to find two linearly independent solutions of equation (4). How does each of these solutions behave near $x = 0$? Find the radius of convergence of the series solutions.
(d) How does the solution in (a) compare with the series solution in (c)
Problem 7 (Do not submit)–Chebyshev polynomials: Consider the Chebyshev equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0, \tag{5}$$

where α is a constant. The solutions to this equation are the famous Chebyshev polynomials that are used in the approximation theory and polynomial interpolation.
a. Show that $x = 1$ and $x = -1$ are regular singular points of equation (5).
b. Find the exponents at each of these singularities (the roots of the indicial equation).
c. Find two linearly independent solutions about $x = 1$. (*Hint:* Write $(1 - x^2) = -(x-1)(x+1) = -(x-1)(2 + (x-1))$ and $x = 1 + (x-1)$ or make the change of variable $x - 1 = t$.)
d. Explain how you write a power series solution about $x = 0$ to find two linearly independent solutions. You must not try to compute the series solution itself.

Problem 8 (Do not submit): Consider the differential equation

$$(\ln x)y'' + \frac{1}{2}y' + y = 0 \tag{6}$$

a. Show that equation 6 has a regular singular point at $x = 1$.
b. Determine the roots of the indicial equation at $x = 1$.
c. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x-1)^{n+r}$ corresponding to the largest root.

problem 1:

$$\left(1+x^{3}\right) y''-6xy=0,$$

expansion point: $x_0=0$. $x_0=0$ is an ordinary point, hence assume power series solution.

$$y=\sum_{n=0}^{\infty} a_n x^n,\ y'=\sum_{n=1}^{\infty} a_n n x^{n-1},\ y''=\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Substitute into ODE:

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}+\sum_{n=2}^{\infty} a_n n(n-1) x^{n+1}-6 \sum_{n=0}^{\infty} a_n x^{n+1}=0$$

Shift the index of the first summation

$$\sum_{m=-1}^{\infty} a_{m+3}(m+3)(m+2) x^{m+1}+\sum_{m=2}^{\infty} a_m m(m-1) x^{m+1}-6 \sum_{m=0}^{\infty} a_m x^{m+1}=0$$

Combining the series, we obtain

$$a_2 \cdot 2 \cdot 1 \cdot x^0+a_3 \cdot 3 \cdot 2 \cdot x^1+a_4 \cdot 4 \cdot 3 \cdot x^2-6 a_0 x-6 a_1 x^2+\sum_{m=2}^{\infty}\left[a_{m+3}(m+3)(m+2)+a_m(m-1)-6 a_m\right] x^{m+1}=0$$

Equating the coefficients of x powers to zero.

$$x^0] \quad 2a_2=0 \rightarrow a_2=0; \qquad x^1] \quad 6a_3=6a_0 \rightarrow a_3=a_0; \qquad x^2] \quad 12a_4=6a_1 \rightarrow a_4=\frac{a_1}{2}$$

$$x^{m+1}] \quad a_{m+3}=\frac{-(m(m-1)-6) a_m}{(m+3)(m+2)},\ m=2,3,\cdots$$

$$a_{m+3}=\frac{-\left(m^2-m-6\right) a_m}{(m+3)(m+2)}=\frac{-(m-3)(m+2) a_m}{(m+3)(m+2)}$$

$$a_{m+3}=\frac{-(m-3) a_m}{(m+3)}$$

$$m=2: \quad a_5-\frac{a_2}{4}=0$$

$$m=3: \quad a_6=\frac{-(0) a_3}{6}=0$$

$$m=4: \quad a_7=\frac{-1 a_4}{7}=\frac{-a_1}{14}$$

$$m=5: a_8=\frac{-2 a_5}{8}=0 \quad \rightarrow \quad a_{11}=a_{14}=a_{17}=0$$

$$m=6: a_9=\frac{-3 a_6}{9}=0 \rightarrow a_{12}=a_{15}=0$$

$$m=7: \quad a_{10}=\frac{-4 a_7}{10}=\frac{4 a_1}{10.14}=\frac{a_1}{35}$$

$$m=8: a_{11}=0$$

$$m=9: a_{12}=0$$

$$m=10: a_{13}=\frac{-7 a_{10}}{13}=\frac{-7 a_1}{13.35}=\frac{-a_1}{65}$$

$$y(x)=a_0+a_0 x^3+a_1 x+\frac{a_1 x^4}{2}-\frac{a_1}{14} x^7+\frac{a_1}{35} x^{10} -\frac{a_1}{65} x^{13}+\cdots$$

$$\boxed{y(x)=a_6\left(1+x^3\right)+a_1\left(x+\frac{x^4}{2}-\frac{x^7}{14}+\frac{x^{10}}{35}-\frac{x^{13}}{65}+\cdots\right)}$$

Radius of Convergence:

$\lim_{m\to\infty}\left|\frac{a_{m+3}x^{m+3}}{a_m x^m}\right| = \lim_{m\to\infty}\left|\frac{a_{m+3}}{a_m}\right|\left|x^3\right|$

$$= \lim_{m\to\infty}\left|\frac{m-3}{m+3}\right|\left|x^3\right| = \left|x^3\right| < 1$$

$$-1 < x < 1$$

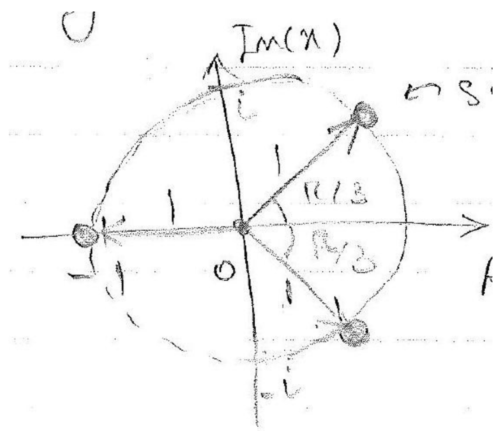
Singular points: $1 + x^3 = 0$

$$\begin{aligned} 1 + x = 0 \quad & i(2n+1)\pi \\ x_s = (-1)^{1/3} \quad & = e \\ s \quad & = \cos\left(\tfrac{(2n+1)\pi}{3}\right) + i\sin\left(\tfrac{2n+1\pi}{3}\right) \end{aligned}$$

The distance from $x_0 = 0$ to x_S is:

$$\cos^2\left(\frac{(2n+1)\pi}{3}\right) + \sin^2\left(\frac{(2n+1)\pi}{3}\right) = 1,$$

which is a lower bound guess for the radius of convergence.



problem 2:

$$\left(1+x^2\right)y'' + xy' - y = 0 \quad , x_0 = 0$$

$x_0 = 0$ is an ordinary point, so use a normal power series.

$$y = \sum_{n=0}^{\infty} a_n x^n,\ y' = \sum_{n=1}^{\infty} a_n n x^{n-1},\ y'' = \sum_{n=2}^{\infty} a_n (n-1) x^{n-2}$$

Substitute into ODE:

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^n + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} a_n x^n$$

shift index of the first summation

$$\sum_{m=n}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{m=2}^{\infty} a_m m(m+1)x^m + \sum_{m=1}^{\infty} a_m m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$a_2 \cdot 2 \cdot 1 \cdot x^0 + a_3 \cdot 3 \cdot 2 \cdot x^1 + a_1 x^1 - a_0 x^0 - a_1 x^1$$
$$+ \sum_{m=2}^{\infty} \left[a_{m+2}(m+2)(m+1) + a_m m(m-1) + a_m \cdot m - a_m \right] x^m = 0$$

$$x^0] \quad 2a_2 - a_0 = 0 \to a_2 = \frac{a_0}{2}; \quad x^1] \quad 6a_3 = 0 \quad \to a_3 = 0$$

$$x^m] \quad a_{m+2}(m+2)(m+1) + a_m \left(m^2 - m + m - 1\right) = 0, \quad m \geqslant 2$$

$$a_{m+2} = \frac{-a_m \left(m^2 - 1\right)}{(m+2)(m+1)} = \frac{-a_m(m-1)}{(m+2)}$$

$$m = 2 : a_4 = \frac{-a_2 \cdot 1}{4} = \frac{-a_0}{4.2} = \frac{-a_0}{8}$$

$$m = 3 : a_5 = \frac{-a_3 \cdot 2}{5} = 0 \quad (a_7 = a_9 = a_{11} = \cdots)$$

$$m = 4 : a_6 = \frac{-a_4 \cdot 3}{6} = \frac{a_0}{2 \cdot 8} = \frac{a_0}{16}$$

$$m = 5 : a_7 = 0$$

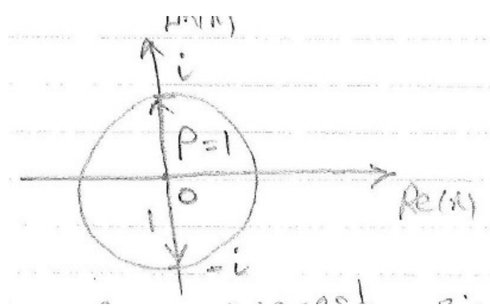
$$m = 6 : a_8 = \frac{-a_6 \cdot 5}{7} = -\frac{a_0 \cdot 5}{16.7}$$

$$\boxed{\begin{aligned} y(x) &= \frac{a}{0} + a_1 x + \frac{a_0}{2} x^2 - \frac{a_0}{8} x^4 + \frac{a_0}{16} x^6 - \frac{a_0 5}{16.7} x^8 + \\ &= a_1 x + a_0 \left(1 + \frac{1}{2} x^2 - \frac{1}{8} x^4 + \frac{1}{16} x^6 - \frac{5}{16.7} x^8 + \cdots\right) \end{aligned}}$$

Radius of Convergence:

ratio test: $\lim_{m \to \infty} \left| \frac{a_{m+2} x^{m+2}}{a_m x^m} \right| = \lim_{m \to \infty} \left| \frac{(m-1)^2}{(m+2)(m+1)} \right| \left| x^2 \right|$

$$\left| x^2 \right| < 1$$
$$-1 < x < 1, \quad \rho = 1$$



$x_0 = 0$ nearest singular points: $x_s = \pm i$
distance from x_0 to $x_s = 1$.

c) $y(0) = 1 \to y(0) = a_0 = 1$

$$y'(0) = 0, \quad y'(0) = a_1 = 0$$

So, the solution is:

$$\boxed{y(x) = 1 + \frac{1}{2} x^2 - \frac{1}{8} x^4 + \frac{1}{16} x^6 + \cdots}$$

Problem 3:

4. $Ly = y'' - \sin x\, y = y'' - (x - x^3/3! + x^5/5! - \cdots)\, y = 0$ 3/

$x = 0$ IS AN ORDINARY POINT SO ASSUME $y(x) = \sum_{n=0} a_n x^n$

$Ly = y'' \quad - \quad \sin x \quad \times \quad y$

$= \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - (x - x^3/3! + \cdot) \sum_{n=0} a_n x^n = 0$

$= a_2 2.1 x^0 + a_3 3.2 x + a_4 4.3 x^2 + a_5 5.4 x^3$

$- (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$

$= 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + 30 a_6 x^4 + \cdots$

$- a_0 x + a_0 \frac{x^3}{6} - a_0 \frac{x^5}{120} + \cdots - a_1 x^2 + a_1 \frac{x^4}{6} - a_1 \frac{x^6}{120} + \cdots$

$- a_2 x^3 + a_2 \frac{x^5}{6} - a_2 \frac{x^7}{120} + \cdots$

GROUP POWERS IN X

$= 2a_2 1 + \{6a_3 - a_0\} x + \{12 a_4 - a_1\} x^2 + \{20 a_5 + \frac{a_0}{6} - a_2\} x^3$

$+ \{30 a_6 - a_3 + \frac{a_1}{6}\} x^4 \cdots$

1] $2a_2 = 0 \Rightarrow a_2 = 0$

x] $6a_3 - a_0 = 0 \quad a_3 = a_0/6$

x^2] $12 a_4 - a_1 = 0 \quad a_4 = a_1/12$

x^3] $20 a_5 + \frac{a_0}{6} - a_2 = 0 \quad a_5 = -\frac{a_0}{120}$

x^4] $30 a_6 = a_3 - \frac{a_1}{6} \qquad a_6 = -\frac{a_1}{180} + \frac{a_3}{30} = \frac{a_0}{180} - \frac{a_1}{180}$

THE TWO SOLUTIONS ARE

$y_0(x) = a_0 \left[1 + \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^6}{180} + \cdots\right]$

$y_1(x) = a_1 \left[x + \frac{x^4}{12} - \frac{x^6}{180} + \cdots\right]$

Problem 4–Frobenius series:

Consider Gauss's hypergeometric equation

$$x(1-x)\,y'' + \big(c-(a+b+1)x\big)\,y' - a\,b\,y = 0. \tag{1}$$

(a) **Classification of singular points.**

Write (1) in standard form

$$y'' + P(x)\,y' + Q(x)\,y = 0, \quad P(x) = \frac{c-(a+b+1)x}{x(1-x)}, \quad Q(x) = -\frac{a\,b}{x(1-x)}.$$

At $x = 0$:

$$\lim_{x\to 0} x\,P(x) = \lim_{x\to 0} \frac{c-(a+b+1)x}{1-x} = c, \quad \lim_{x\to 0} x^2\,Q(x) = 0,$$

both finite. Hence $x = 0$ is a *regular* singular point.

At $x = 1$:

$$\lim_{x\to 1}(x-1)\,P(x) = \lim_{x\to 1}(x-1)\,\frac{c-(a+b+1)x}{x(1-x)} = \lim_{x\to 1}\frac{-(c-(a+b+1)x)}{x} = -(c-(a+b+1)),$$

$$\lim_{x\to 1}(x-1)^2\,Q(x) = \lim_{x\to 1}(x-1)^2\Big(-\tfrac{ab}{x(1-x)}\Big) = 0,$$

both finite. Thus $x = 1$ is also a *regular* singular point.

At $x = \infty$: Set $x = 1/z$, $y(x) = Y(z)$. Then

$$\frac{dy}{dx} = -z^2 Y', \quad \frac{d^2y}{dx^2} = z^4 Y'' + 2z^3 Y',$$

Substitute into the ODE:

$$\frac{1}{z}\Big(1-\tfrac{1}{z}\Big)\big(z^4 Y'' + 2z^3 Y'\big) + \Big(c-(a+b+1)\tfrac{1}{z}\Big)\big(-z^2 Y'\big) - ab\,Y = 0.$$

Multiply by z^2 and simplify:

$$z^2(z-1)\,Y'' + \big((2-c)z^2 + (a+b-1)z\big)\,Y' - ab\,Y = 0.$$

You can show that $z = 0$, hence $x = \infty$ is a *regular* singular point.

The only singular points are $x = 0, 1, \infty$; all other $x \in (0,1)$ or $(1,\infty)$ are ordinary.

(b) **Exponents at the regular singular points.**

We seek a Frobenius solution $y = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$.

At $x = 0$: Substitute $y = x^r \sum a_n x^n$ into the ODE and collect the lowest power x^{r-1}:

$$x(1-x)\,y'' \sim r(r-1)x^{r-1}, \quad (c-(a+b+1)x)\,y' \sim c\,r\,x^{r-1}.$$

Indicial equation (You can plug in the limits into the general indicial equation):

$$r(r-1) + c\,r = 0 \implies r\,(r+c-1) = 0 \quad \implies \quad r_1 = 0, \quad r_2 = 1-c.$$

At $x = 1$: Let $t = x-1$, set $y = t^r \sum b_n t^n$.
We get

$$r(r-1) + (a+b+1-c)\,r = 0 \implies r\,(r+c-a-b-1) = 0 \quad \implies \quad r_1 = 0,\ r_2 = c-a-b.$$

At $x = \infty$: From the transformed ODE in z, the indicial exponents at $z = 0$ are

$$r_1 = a, \quad r_2 = b.$$

(c) Assuming that $1-c$ is not a positive integer, in the neighborhood of $x=0$ one solution is

$$y_1(x) = 1 + \frac{a\,b}{c\cdot 1!}\,x + \frac{a(a+1)\,b(b+1)}{c(c+1)\,2!}\,x^2 + \cdots .$$

By the ratio test (or from the singularities at $x=0,1$), the radius of convergence is

$$R = 1.$$

(d) Assuming that $1-c$ is not an integer or zero, a second linearly independent solution on $0<x<1$ is

$$y_2(x) = x^{1-c}\left[1+\frac{(a-c+1)(b-c+1)}{(2-c)\,1!}\,x+\frac{(a-c+1)(a-c+2)\,(b-c+1)(b-c+2)}{(2-c)(3-c)\,2!}\,x^2+\cdots\right].$$

Problem 5

(a) Let $t=\frac{1}{x}$. Note

$$\begin{aligned}\frac{dy}{dx} &= \frac{dt}{dx}\frac{dy}{dt} = -t^2\frac{dy}{dt},\\ \frac{d^2y}{dx^2} &= \frac{dt}{dx}\frac{d}{dt}(\frac{dy}{dx})\\ &= -t^2\frac{d}{dt}(-t^2\frac{dy}{dt})\\ &= t^4\frac{d^2y}{dt^2}+2t^3\frac{dy}{dt}.\end{aligned}$$

Substituting this into the ODE and simplifying, we have

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0.$$

$t=0$ is a singular point of this new equation. Since $t\cdot(-\frac{2}{t})$ and $t^2\cdot(\frac{2}{t^2})$ are both finite at $t=0$, $t=0$ is a regular singular point. This implies $x=\infty$ is a regular singular point.

Using the expansions of $t\cdot(-\frac{2}{t})$ and $t^2\cdot(\frac{2}{t^2})$ and the formula for the indicial equation $m(m-1)+mp_0+q_0$, we deduce $m=2,1$, which are also the indices corresponding to $x=\infty$.

(b) By the change of variables formulae outlined in (a), we have the following transformed ODE

$$y'' + \left(\frac{2}{t} - \frac{ct-1}{t^2}\right)y' + \frac{a}{t^3}y = 0.$$

Check that $t\cdot(\frac{2}{t}-\frac{ct-1}{t^2})$ and $t^2\cdot\frac{a}{t^3}$ both tend to ∞ as $t\to 0$. Hence the singularity at $t=0$ (and correspondingly $x=\infty$) is irregular.

Problem 6

See lecture notes Section 2.3.3, page 51

Problem 7:

The Chebyshev equation,

$$(1-x^2)\,y'' - xy' + \alpha^2 y = 0$$

Singular points, $x = \pm 1$

a. $$\lim_{x\to 1} (x-1)\frac{-x}{(1-x^2)} = \lim_{x\to 1} \frac{+x}{1+x} = \frac{1}{2} = p_0$$ finite

$$\lim_{x\to 1} (x-1)^2 \frac{\alpha^2}{(1-x^2)} = \lim_{x\to 1} \frac{-\alpha^2 (x-1)}{(x+1)} = 0 = q_0$$

$x_0 = 1$ regular singular point

$$\lim_{x\to -1} (x+1)\frac{-x}{(1-x^2)} = \lim_{x\to -1} \frac{-x}{(1-x)} = \frac{1}{2} = p_0$$

$$\lim_{x\to -1} (x+1)^2 \frac{\alpha^2}{(1-x^2)} = \lim_{x\to -1} \frac{\alpha^2 (x+1)}{(1-x)} = 0 = q_0$$

$x_0 = -1$ is also a regular singular point

b.

The indicial equation,

$$r(r-1) + p_0 r + q_0 = 0 \Rightarrow r(r-1) + \frac{1}{2} r + 0 = 0$$

$$\Rightarrow r(r - 1/2) = 0 \Rightarrow r_{1,2} = 0, 1/2$$

c. $x_0 = 1$, $y = \sum_{n=0}^{\infty} a_n (x-1)^{n+r}$

$$y' = \sum_{n=0}^{\infty} a_n (n+r)(x-1)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-1)^{n+r-2}$$

$$Ly = (1-x^2)y'' - xy' + \alpha^2 y = 0$$

$$= -2(x-1)y'' - (x-1)^2 y'' - y' - (x-1)y' + \alpha^2 y = 0$$

$$= -2 \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-1)^{n+r-1} \longrightarrow \begin{cases} n+r-1 = m+r \\ n-1 = m \\ n = m+1 \\ n=0 \to m=-1 \end{cases}$$

$$- \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-1)^{n+r} \longrightarrow n = m$$

$$- \sum_{n=0}^{\infty} a_n (n+r)(x-1)^{n+r-1} \longrightarrow \boxed{\begin{matrix} n = m+1 \\ n=0 \to m=-1 \end{matrix}}$$

$$- \sum_{n=0}^{\infty} a_n (n+r)(x-1)^{n+r} \qquad n=m$$

$$+ \alpha^2 \sum_{n=0}^{\infty} a_n (x-1)^{n+r} \qquad n=m$$

$$= 0$$

Peel off the $m=-1$ terms.

$$-2a_0 r(r-1)(x-1)^{r-1} - a_0 r(x-1)^{r-1}$$

$$+\sum_{m=0}^{\infty}\Big[-2a_{m+1}(m+r+1)(m+r) - a_m(m+r)(m+r-1)$$

$$- a_{m+1}(m+r+1) - a_m(m+r) + \alpha^2 a_m\Big]x^{m+r} = 0$$

$x^{r-1}]$ $\quad a_0[-2r(r-1) - r] = 0$

$$2r^2 - 2r + r = 0 \rightarrow 2r^2 - r = 0,\ r(r-1/2) = 0$$

$$r_{1,2} = 0, 1/2$$

$x^{m+r}]$, $m \geq 0$ $\quad a_{m+1}\left[-(m+r+1)(2m+2r+1)\right]$

$$= a_m\left[(m+r)(m+r) - \alpha^2\right]$$

$$a_{m+1} = \frac{-a_m\left[(m+r)^2 - \alpha^2\right]}{(m+r+1)(2m+2r+1)}$$

$\underline{r_1 = 1/2:}$ $\quad a_{m+1} = \dfrac{-a_m\left[m^2 + m + 1/4 - \alpha^2\right]}{(m+3/2)(2m+2)}, \quad m \geq 0$

$$a_1 = \frac{-a_0(1/4 - \alpha^2)}{3} = \frac{a_0}{12}(3\alpha - 1)$$

$$a_2 = \frac{-a_1\left[9/4 - \alpha^2\right]}{5/2 \cdot 4} = \frac{a_0(9/4 - \alpha^2)(1 - 3\alpha^2)}{120}$$

$\vdots$

$$y_1(x) = a_0(x-1)^{1/2}\left[1 + \frac{(3\alpha - 1)}{12}(x-1) + \frac{(9/4 - \alpha^2)(1 - 3\alpha^2)}{120}(x-1)^2 + \cdots\right]$$

$\underline{r_2 = 0:}$ $\quad a_{m+1} = \dfrac{-a_m(m^2 - \alpha^2)}{(m+1)(2m+1)}$, $m \geq 0$

$$a_1 = \frac{+a_0 \cdot \alpha^2}{1} = \alpha^2 a_0$$

$$a_2 = \frac{-a_1(1-\alpha^2)}{2.3} = \frac{a_0 \alpha^2(\alpha^2-1)}{6}$$

$$a_3 = \frac{-a_2(4-\alpha^2)}{3.5} = \frac{a_0 \alpha^2(\alpha^2-1)(\alpha^2-4)}{6.5.3}$$

$$y_2(x) = a_0(x-1)^0\left[1 + \alpha^2(x-1) + \frac{\alpha^2(\alpha^2-1)}{6}(x-1)^2 + \frac{\alpha^2(\alpha^2-1)(\alpha^2-4)}{6.5.3}(x-1)^3 + \cdots\right]$$

Note: If α is a nonnegative integer n, then the series terminates and this solution becomes a polynomial of degree n. $\rightarrow$ Chebyshev Polynomials

d. $x = 0$ is an ordinary point, so the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$, should yield two linearly independent solutions.

Problem 8:

$$(\ln x)\, y'' + \frac{1}{2} y' + y = 0$$

a. $\ln x = 0$ at $x_0 = 1 \rightarrow x_0$ is a singular point.

$$\lim_{x\to 1} (x-1) \cdot \frac{1}{2\ln x} = \lim_{x\to 0} \frac{1}{2/x} = \frac{1}{2} = p_0, \text{ finite}$$

(L'hopital's rule)

$$\lim_{x\to 1} (x-1)^2 \cdot \frac{1}{2\ln x} = \lim_{x\to 1} \frac{2(x-1)}{2/x} = 0 = q_0, \quad \text{finite}$$

$x_0 = 1$ is a regular singular point.

b. The indicial equation:

$$r(r-1) + r/2 = 0 \rightarrow r^2 - r/2 = 0 \rightarrow r_{1,2} = 0, 1/2$$

the roots

c.
$$y = \sum_{n=0}^{\infty} a_n (x-1)^{n+r}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r)(x-1)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)(x-1)^{n+r-2}$$

Taylor expand $\ln x$ about $x = 1$:

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$Ly = \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots\right]y'' + \frac{1}{2}y' + y = 0 \quad (1)$$

Now expand the Frobenius solution:

$r = 1/2$:

$$y = a_0(x-1)^{1/2} + a_1(x-1)^{3/2} + a_2(x-1)^{5/2} + a_3(x-1)^{7/2} + \cdots$$

$$y' = a_0 \cdot \tfrac{1}{2}(x-1)^{-1/2} + a_1 \frac{3}{2}(x-1)^{1/2} + a_2 \cdot \frac{5}{2}(x-1)^{3/2} + a_3 \cdot \tfrac{7}{2}(x-1)^{5/2} + \cdots$$

$$y'' = a_0\left(-\frac{1}{4}\right)(x-1)^{-3/2} + a_1 \cdot \frac{3}{4}(x-1)^{-1/2} + a_2 \cdot \frac{15}{4}(x-1)^{1/2} + \cdots$$

Now substitute in (1), collect all coefficients of similar powers (needs a bit of algebra ...):

$$y_1(x) = a_0\left[(x-1)^{1/2} - \frac{3}{4}(x-1)^{3/2} + \frac{53}{480}(x-1)^{5/2} + \cdots\right]$$