Math 257/316 Assignment 3, 2025

Due Monday June 2 Submit online only the final solution of Problem 1 in a PDF document (or direct entry) on Canvas by 11:59 pm of the due date

Problem 1 (Submit only the final solution)– Mixed BC: Apply the method of separation of variables to determine the solution to the one dimensional heat equation with the following Mixed homogeneous boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi/2, \quad t > 0$$

$$\text{BC} : u(0,t) = 0 = \frac{\partial u(\pi/2,t)}{\partial x}$$

$$\text{IC} : u(x,0) = \sin(5x)$$

Solution ***1. Separation of variables.*** *Assume*

$$u(x,t) = X(x)\,T(t).$$

Substitute into the PDE to get

$$X(x)\,T'(t) = X''(x)\,T(t) \implies \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

2. Temporal ODE.

$$T'(t) + \lambda\,T(t) = 0 \quad \implies \quad T(t) = e^{-\lambda\,t}.$$

3. Eigenvalue problem.

$$X''(x) + \lambda\,X(x) = 0, \quad X(0) = 0, \quad X'\big(\tfrac{\pi}{2}\big) = 0.$$

The general solution is

$$X(x) = A\cos(\mu x) + B\sin(\mu x), \quad \lambda = \mu^2.$$

The boundary $X(0) = 0$ *gives* $A = 0$*, so* $X(x) = B\sin(\mu x)$*. Then*

$$X'\big(\tfrac{\pi}{2}\big) = B\,\mu\cos\!\big(\mu\pi/2\big) = 0 \implies \cos\!\big(\mu\pi/2\big) = 0 \implies \mu\pi/2 = (2n{+}1)\frac{\pi}{2},\ n = 0,1,2,\ldots$$

Hence the eigenvalues and eigenfunctions are

$$\mu_n = 2n+1, \quad \lambda_n = (2n+1)^2, \quad X_n(x) = \sin\!\big((2n+1)x\big).$$

Show that the cases $\lambda < 0$ and $\lambda = 0$ give trivial solutions.

4. General solution.

$$u(x,t) = \sum_{n=0}^{\infty} A_n \sin\big((2n+1)x\big)\, e^{-(2n+1)^2\, t}.$$

5. Determine coefficients from the initial condition. *At* $t = 0$,

$$u(x,0) = \sin(5x) = \sum_{n=0}^{\infty} A_n \sin\big((2n+1)x\big).$$

But $\sin(5x) = \sin\big((2 \cdot 2 + 1)x\big)$, *so the only nonzero coefficient is* $A_2 = 1$ *and* $A_n = 0$ *for* $n \neq 2$. **Note that here I used orthogonality property**

$$\int_{-L}^{L} \sin\Big(\tfrac{(2n+1)\pi x}{2L}\Big)\, \sin\Big(\tfrac{(2m+1)\pi x}{2L}\Big)\, dx = \begin{cases} L, & n = m, \\ 0, & n \neq m, \end{cases}$$

$$\int_{-L}^{L} \cos\Big(\tfrac{(2n+1)\pi x}{2L}\Big)\, \cos\Big(\tfrac{(2m+1)\pi x}{2L}\Big)\, dx = \begin{cases} L, & n = m, \\ 0, & n \neq m, \end{cases}$$

or

$$\int_{0}^{L} \sin\Big(\tfrac{(2n+1)\pi x}{2L}\Big)\, \sin\Big(\tfrac{(2m+1)\pi x}{2L}\Big)\, dx = \begin{cases} \frac{L}{2}, & n = m, \\ 0, & n \neq m, \end{cases}$$

$$\int_{0}^{L} \cos\Big(\tfrac{(2n+1)\pi x}{2L}\Big)\, \cos\Big(\tfrac{(2m+1)\pi x}{2L}\Big)\, dx = \begin{cases} \frac{L}{2}, & n = m, \\ 0, & n \neq m. \end{cases}$$

6. Final solution.

$$\boxed{u(x,t) = \sin(5x)\, e^{-25\, t}.}$$

Problem 2 (Do not submit) Find all eigenvalues and corresponding eigenfunctions for the following eigenvalue problem
DE:

$$y'' + \lambda y = 0, \quad 0 < x < L.$$

Boundary Conditions:

$$y'(0) = 0, \quad y(L) = 0.$$

Please show all the cases when solving the eigenvalue problem.

Solution *We look for nontrivial solutions $y(x)$ and corresponding values of λ. Three cases are: $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.*

Case 1: $\lambda = 0$**.** *Show that there are no nontrivial solutions for $\lambda = 0$.*

Case 2: $\lambda < 0$**.** *Show that it also gives a triavial solution $y \equiv 0$.*

Case 3: $\lambda > 0$**.** *Let $\lambda = \mu^2$ with $\mu > 0$. The ODE becomes*

$$y'' + \mu^2\, y = 0,$$

with the general solution

$$y(x) = A\cos(\mu x) \; + \; B\sin(\mu x).$$

$$y'(x) \; = \; -A\,\mu\,\sin(\mu x) \; + \; B\,\mu\,\cos(\mu x),$$

using $y'(0) = 0$.
We have

$$y'(0) \; = \; -A\,\mu \cdot 0 \; + \; B\,\mu \cdot 1 \; = \; B\,\mu \; = \; 0 \implies B = 0,$$

hence,

$$y(x) = A\,\cos(\mu x).$$

$y(L) = 0$ gives

$$A\,\cos(\mu L) \; = \; 0 \implies \cos(\mu L) = 0.$$

Hence

$$\mu L \; = \; \Big(2n+1\Big)\,\frac{\pi}{2}, \quad n = 0, 1, 2, \ldots \implies \mu_n \; = \; \frac{(2n+1)\,\pi}{2L}.$$

Therefore the eigenvalues are

$$\lambda_n \; = \; \mu_n^2 \; = \; \left(\tfrac{(2n+1)\,\pi}{2L}\right)^2, \quad n = 0, 1, 2, \ldots$$

The corresponding eigenfunctions are

$$y_n(x) \; = \; \cos\Big(\mu_n\, x\Big) \; = \; \cos\Big(\tfrac{(2n+1)\,\pi}{2L}\, x\Big), \quad n = 0, 1, 2, \ldots$$

Problem 3 (Do not submit): Consider the following initial boundary value problem for the heat equation:

$$
\begin{aligned}
&\text{P.D.E.} \qquad u_t + u = 4u_{xx}, \;\; 0 < x < \pi/2, \;\; t > 0\\
&\text{B.C.} \qquad u_x(0,t) = 0, \;\; \text{and} \;\; u(\pi/2,t) = 0\\
&\text{I.C.} \qquad u(x,0) = \begin{cases} 1 & \text{if } 0 < x < \frac{\pi}{4}, \\ 0 & \text{if } \frac{\pi}{4} < x < \frac{\pi}{2}. \end{cases}
\end{aligned}
$$

(a) Apply the method of separation of variables to determine the solution of the heat equation (above) together with the given mixed boundary conditions (It is not necessary to give all the details of the different cases for the eigenvalue problem).

(b) (i) Use the given initial condition to determine Fourier coefficients. Sketch the extension of the initial condition you would assume on the interval $-2\pi \le x \le 2\pi$.

(b) (ii) Use Parseval's identity (or ortherwise) to find the series for π^2 (See the formula sheet for Parseval's identity).

Hint: It may be useful to know that:

$$\sin^2\left(\frac{(2n+1)\pi}{2}\right) = 1/2$$

Solution ***(a) Separation of variables.*** *Assume* $u(x,t) = X(x)\,T(t)$. *Substitution gives*

$$X\,T' + X\,T = 4\,X''\,T \implies \frac{T'}{T} + 1 = 4\,\frac{X''}{X} = -4\lambda.$$

Thus

$$X'' + \lambda\,X = 0, \qquad T' + (1 + 4\lambda)\,T = 0.$$

The boundary conditions $X'(0) = 0$, $X(\pi/2) = 0$ *admit nontrivial solutions for*

$$\lambda_n = (2n+1)^2, \quad X_n(x) = \cos\big((2n+1)x\big), \quad n = 0, 1, 2, \ldots$$

and

$$T_n(t) = \exp\big[-(1 + 4(2n+1)^2)\,t\big].$$

Hence the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\big((2n+1)x\big)\, e^{-(1+4(2n+1)^2)\,t}.$$

(b)(i) Using initial condition. *At* $t = 0$ *we have*

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\big((2n+1)x\big) \quad (0 < x < \tfrac{\pi}{2}).$$

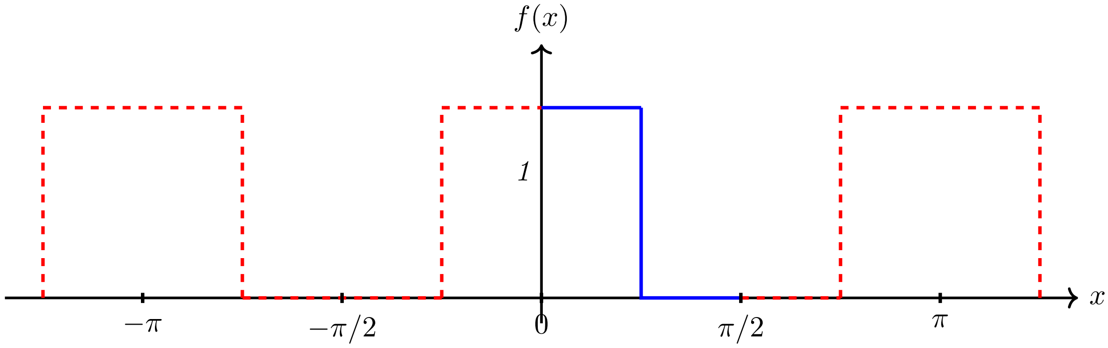
Using the orthogonality,

$$A_n = \frac{2}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} f(x) \cos\big((2n+1)x\big)\, dx$$

$$= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \cos\big((2n+1)x\big)\, dx$$

$$= \frac{4}{\pi\,(2n+1)} \sin\Big(\tfrac{(2n+1)\pi}{4}\Big).$$

We assume an even extension of the function in the interval



(b)(ii) Parseval's identity and series for π^2.

$$\frac{4}{\pi} \int_0^{\pi/2} f(x)^2\, dx = 1 = \sum_{n=0}^{\infty} A_n^2.$$

Hence

$$\sum_{n=0}^{\infty} \frac{8}{\pi^2 (2n+1)^2} = 1 \implies \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

Therefore

$$\boxed{\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.}$$

Problem 4 (Do not submit): (a) Briefly describe how you would use the method of finite differences to obtain an approximate solution of the initial boundary value problem in **Problem 3** that is accurate to $O(\Delta^2, \Delta)$ terms. Use the notation $u_n^k = u(x_k, t_k)$ to represent the nodal values in the finite different mesh.

(b) Given that the solution at $k+1$ time can be computed in matrix form by:

$$\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}^{k+1} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}^{k} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}$$

(i) Describe the entries of the coefficient matrix $A = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$ and the constant vector $F = [f_1, f_2, \cdots, f_N]^T$ given the boundary conditions $u_x(0,t) = 1$ and $u(\pi/2, t) = 1$. (use central difference scheme)

Solution *Consider the PDE*

$$u_t + u = 4u_{xx}$$

Discretize in time by forward difference scheme and space by central difference scheme and substitute into the PDE to get

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} + u_n^k = \frac{4\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)}{\Delta x^2}, \; r = \frac{4\Delta t}{\Delta x^2}$$
$$u_n^{k+1} = u_n^k - \Delta t u_n^{k'} + r u_{n+1}^k - 2r u_n^k + r u_{n-1}^k$$
$$u_n^{k+1} = (1 - \Delta t - 2r)u_n^k + r u_{n+1}^k + r u_{n-1}^k$$

Apply central difference scheme on the left boundary

$$\frac{u_{n+1}^k - u_{n-1}^k}{2\Delta x} = 0$$

at $n = 0$*, we have* $\frac{u_1^k - u_{-1}^k}{2\Delta x} = 0$*, which introduces a ghost node*

$$u_0^{k+1} = (1 - \Delta t - 2r)u_0^k + r u_1^k + r \boxed{u_{-1}^k}$$

See Lecturer notes.

The interior nodes form a tridiagonal matrix with the main diagonal being $1-\Delta t-2r$ *and off diagonal* r *and* r.
Show that

$$A = \begin{bmatrix} 1-\Delta t-2r & 2r & 0 & 0 & \cdots \\ r & 1-\Delta t-2r & r & 0 & \cdots \\ 0 & r & 1-\Delta t-2r & r & \cdots \\ \vdots & & & & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -2r\Delta x \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Problem 5 (Do not submit): Use Taylor's series expansions about the point x for $f(x-\Delta x)$, and $f(x-2\Delta x)$ to find a backward finite difference approximation for $f'(x)$ that has a second order accuracy.

Solution *Consider the Taylor expansion;*

$$\textit{(1)} \quad f(x) = f(x)$$

$$\textit{(2)} \quad f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) - \frac{\Delta x^3}{3!} f'''(x) + O\left(\Delta x^4\right)$$

$$\textit{(3)} \quad f(x-2\Delta x) = f(x) - 2\Delta x f'(x) + \frac{4\Delta x^2 f''(x)}{2} - \frac{8\Delta x^3 f'''(x) + O\left(\Delta x^4\right)}{3!}$$

To find an approximation for $f'(x)$ *with second order accuracy, we need to have:*
$A \cdot (1) + B \cdot (2) + C \cdot (3) = f'(x) + O\left(\Delta x^2\right)$
Find coefficients $A, B,$ *and* C *:*

$$\begin{cases} A+B+C=0 \rightarrow \;\; \textit{Coefficient of } f(x) \textit{ should go to zero} \\ -\Delta x B - 2\Delta x C = 1 \rightarrow \;\; \textit{coefficients of } f'(x) \\ \dfrac{\Delta x^2}{2} \cdot B + \dfrac{4\Delta x^2 c}{2} = 0 \rightarrow \;\; \textit{to ensure second accuracy} \end{cases}$$

Solve to obtain $A = 3/(2\Delta x)$, $B = -2/(\Delta x)$ *and* $C = 1/(2\Delta x)$
The backward finite difference approximation is:

$$f'(x) = \frac{3f(x) - 4f(x-\Delta x) + f(x-2\Delta x)}{2\Delta x} + 0\left(\Delta x^2\right)$$

Problem 6 (Do not submit): Consider the differential equation

$$2x^2 y'' + (2x+1)xy' - y = 0.$$

Identify singular points. Are they regular? Compute the Frobenius series (only the first 3 terms of each independent solution) and determine its radius of convergence.

Solution *Show that* $x = 0$ *is a* regular *singular point.*
Indicial equation. *Set*

$$y(x) \;=\; \sum_{n=0}^{\infty} a_n\, x^{\,n+r}, \quad a_0 \neq 0.$$

Then

$$y' \;=\; \sum_{n=0}^{\infty} a_n\,(n+r)\,x^{\,n+r-1}, \qquad y'' \;=\; \sum_{n=0}^{\infty} a_n\,(n+r)(n+r-1)\,x^{\,n+r-2}.$$

Substitute into the ODE

$$2x^2 \sum_{n=0}^{\infty} a_n(n{+}r)(n{+}r{-}1)\,x^{\,n+r-2} + (2x{+}1)\,x \sum_{n=0}^{\infty} a_n(n{+}r)\,x^{\,n+r-1} - \sum_{n=0}^{\infty} a_n\,x^{\,n+r} \;=\; 0.$$

Simplify term by term:

$$2\sum_{n=0}^{\infty} a_n\,(n{+}r)(n{+}r{-}1)\,x^{\,n+r} + 2\sum_{n=0}^{\infty} a_n\,(n{+}r)\,x^{\,n+r+1} + \sum_{n=0}^{\infty} a_n\,(n{+}r)\,x^{\,n+r} - \sum_{n=0}^{\infty} a_n\,x^{\,n+r} \;=\; 0.$$

Shift the index:

$$2\sum_{n=0}^{\infty} a_n\,(n{+}r)(n{+}r{-}1)\,x^{\,n+r} + 2\sum_{n=1}^{\infty} a_{\,n-1}\,(n{-}1{+}r)\,x^{\,n+r} + \sum_{n=0}^{\infty} a_n\,(n{+}r)\,x^{\,n+r} - \sum_{n=0}^{\infty} a_n\,x^{\,n+r} \;=\; 0.$$

Simplify and Equate coefficients of x-powers to zero

$$2\,r(r-1)\,a_0 + r\,a_0 - a_0 \;=\; 0, \quad \Longrightarrow \quad \big[\,2r^2 - 2r + r - 1\,\big]a_0 = 0 \;\Longrightarrow\; 2r^2 - r - 1 = 0.$$

Solve $2r^2 - r - 1 = 0$ *to obtain*

$$r \;=\; \frac{1 \pm \sqrt{1+8}}{4} \;=\; \frac{1 \pm 3}{4} \;\;\Longrightarrow\;\; r_1 = 1, \quad r_2 = -\tfrac{1}{2}.$$

Recurrence relation. *For* $n \geq 1$*, we have:*

$$2(n+r)(n+r-1)\,a_n \;+\; 2(n-1+r)\,a_{\,n-1} \;+\; (n+r)\,a_n \;-\; a_n \;=\; 0.$$

Simplify to get

$$a_n \;=\; -\,\frac{2\,(n-1+r)}{2(n+r)(n+r-1) \;+\; (n+r) \;-\; 1}\; a_{\,n-1}.$$

or

$$\boxed{a_n = -\,\frac{2\,(n-1+r)}{2(n+r)^2 \;-\; (n+r) \;-\; 1}\; a_{\,n-1}, \quad n \geq 1.}$$

(i) Frobenius series for $r = 1$***.*** *The recurrence relation simplifies to*

$$a_n = -\,\frac{2n}{n(2n+3)}\; a_{\,n-1} = -\,\frac{2}{2n+3}\; a_{\,n-1}, \quad n \geq 1.$$

$a_0 \neq 0$ *is arbitrary*

$$a_1 = -\,\frac{2}{5}\,a_0, \qquad a_2 = -\,\frac{2}{7}\,a_1 = \frac{4}{35}\,a_0, \qquad a_3 = -\,\frac{2}{9}\,a_2 = -\,\frac{8}{315}\,a_0, \;\ldots$$

Thus the first three terms of the solution $y_1(x)$ *are;*

$$y_1(x) = a_0\,x^{\,1} \;+\; a_1\,x^{\,2} \;+\; a_2\,x^{\,3} \;+\; \cdots = a_0\Big[x \;-\; \tfrac{2}{5}\,x^2 \;+\; \tfrac{4}{35}\,x^3 \;+\; \cdots\Big].$$

(ii) Frobenius series for $r = -\frac{1}{2}$***.*** *The recurrence relation simplifies to:*

$$a_n = -\,\frac{2\,(n-\frac{3}{2})}{n(2n-3)}\; a_{\,n-1}, \quad n \geq 1.$$

Choose $a_0 \neq 0$ *arbitrary*

$$a_1 = -\,\frac{2(1-\frac{3}{2})}{1\cdot(2-3)}\,a_0 = -\,\frac{-1}{-1}\,a_0 = -\,a_0,$$

$$a_2 = -\frac{2(2-\frac{3}{2})}{2\cdot(4-3)}\,a_1 = -\frac{2\cdot\frac{1}{2}}{2\cdot 1}(-a_0) = \frac{1}{2}\,a_0,$$

$$a_3 = -\frac{2(3-\frac{3}{2})}{3\cdot(6-3)}\,a_2 = -\frac{2\cdot\frac{3}{2}}{3\cdot 3}\,\frac{a_0}{2} = -\frac{1}{6}\,a_0,\ \ldots$$

$y_2(x)$ becomes

$$y_2(x) = a_0x^{-\frac{1}{2}}\Big(1\ -\ x\ +\ \tfrac{1}{2}\,x^2\ +\ \cdots\Big).$$

Use recurrence relation to show that $R = \infty$