Math 257/316 Assignment 4, 2025

Tuesday June 10, 2025. Submit online in a PDF document on Canvas by 11:59 pm of the due date

Problem 1 (Do not submit) Fourier series - Determine whether the following functions are odd, even or neither:
(a) $f(x) = x^2 + |x|$ (b) $f(x) = e^{\sin^2 x}$ (c) $f(x) = \cosh x + \sinh x$ $f(x) = \tanh x$

Solution

(a) $f(x) = x^2 + |x|$

To check if f is even, odd, or neither, we compute $f(-x)$ and compare it with $f(x)$ and $-f(x)$.

$$f(-x) = (-x)^2 + |-x| = x^2 + |x| = f(x).$$

Since $f(-x) = f(x)$ for all x, the function f is **even**.
(b) $f(x) = e^{\sin^2 x}$

$$f(-x) = e^{\sin^2(-x)} = e^{(\sin(-x))^2} = e^{(-\sin x)^2} = e^{\sin^2 x} = f(x).$$

Since $f(-x) = f(x)$ for all x, f is **even**.
(c) $f(x) = \cosh x + \sinh x$
Recall that $\cosh x$ is even and $\sinh x$ is odd. That is,

$$\cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x.$$

Now,

$$f(-x) = \cosh(-x) + \sinh(-x) = \cosh x - \sinh x.$$

Hence, f is **neither even nor odd**.
(d) $f(x) = \tanh x$
Recall the definition:

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

Therefore,

$$f(-x) = \tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)} = \frac{-\sinh x}{\cosh x} = -\tanh x = -f(x).$$

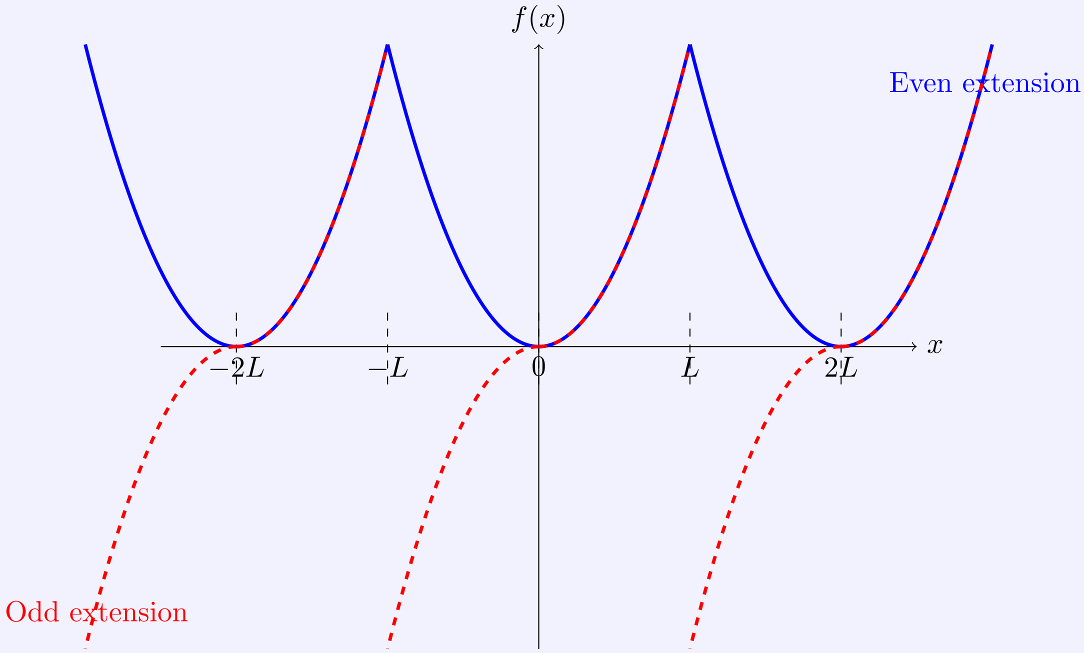
Since $f(-x) = -f(x)$, the function f is **odd**.

Problem 2 (Submit only the sketches for part (b) and (c)) Fourier series - For the following functions, sketch even and odd extensions of $f(x)$ on $[-L, L]$, and find the Fourier sine series of $f(x)$ assuming that the function has a period of $2L$.

(a) $f(x) = x^2, \quad 0 \le x \le L$

Solution

Odd and even extensions over $[-2L, 2L]$



Fourier cosine series

The Fourier cosine series coefficients are unchanged from the single period case:

$$a_0 = \frac{2}{L}\int_0^L x^2 dx = \frac{2L^2}{3},$$

Integration by parts (check to be sure it is correct)

$$a_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{4L^2}{(n\pi)^2}(-1)^n.$$

Hence,

$$\boxed{x^2 = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right).}$$

Fourier sine series

The Fourier sine series coefficients are:

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right) dx = \frac{4L^2}{(n\pi)^3}\big((-1)^n - 1\big).$$

Nonzero only for odd n, that is,

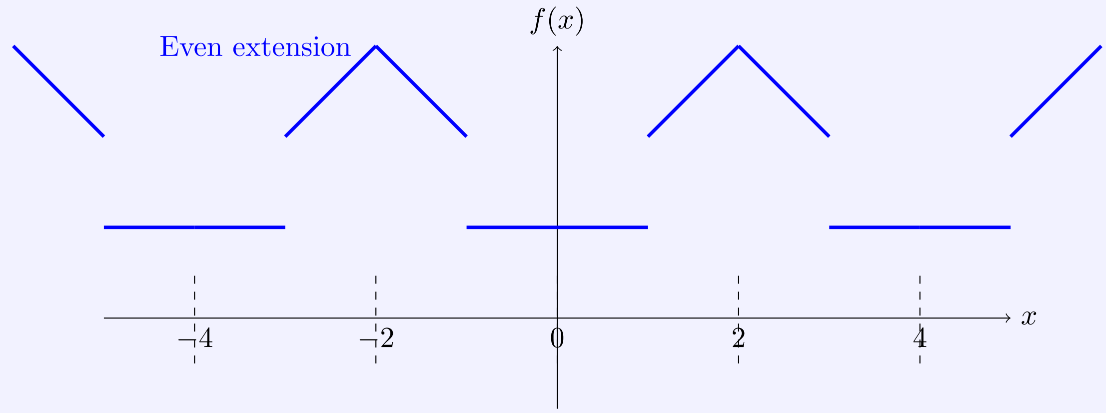
$$b_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -\dfrac{8L^2}{(n\pi)^3}, & \text{if } n \text{ is odd.} \end{cases}$$

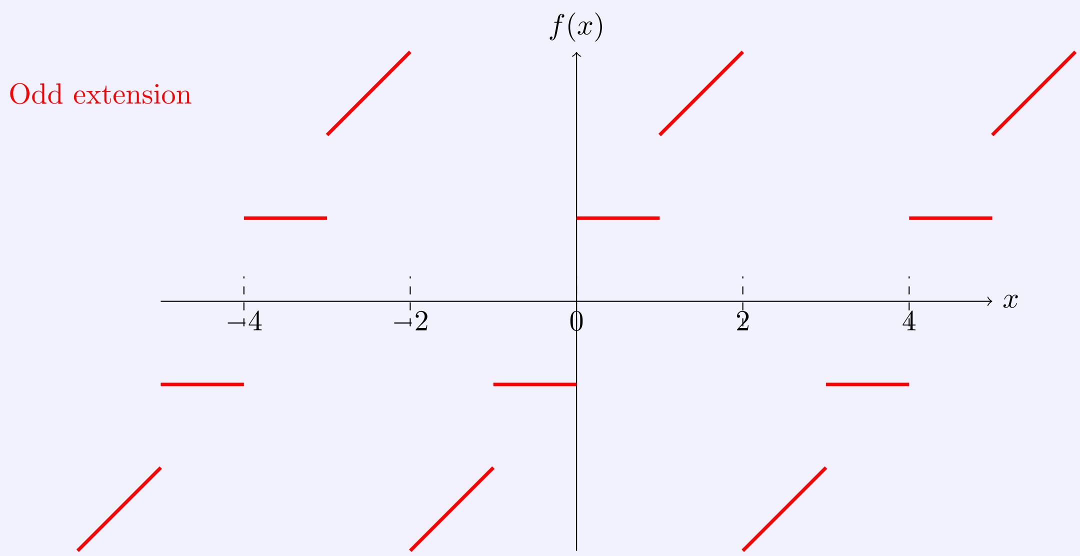
$$\boxed{x^2 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{-8L^2}{(n\pi)^3}\sin\left(\frac{n\pi x}{L}\right) = \sum_{k=0}^{\infty} \frac{-8L^2}{((2k+1)\pi)^3}\sin\left(\frac{(2k+1)\pi x}{L}\right).}$$

(b) $L = 2, \quad f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ x+1 & \text{if } 1 \le x \le 2 \end{cases}$

Solution

Even and Odd Extensions





Fourier Cosine Series

The Fourier cosine series is given by

$$f_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right),$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)dx = \int_0^2 f(x)dx,$$

$$a_n = \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx.$$

$$a_0 = \int_0^1 1\,dx + \int_1^2 (x+1)dx = (1-0) + \left[\frac{x^2}{2} + x\right]_1^2 = 3.5.$$

$$a_n = \int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (x+1)\cos\left(\frac{n\pi x}{2}\right)dx = I_1 + I_2.$$

$$I_1 = \int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

$$I_2 = \int_1^2 (x+1)\cos\left(\frac{n\pi x}{2}\right)dx,$$

Apply integration by parts, Let:

$$u = x+1, \quad dv = \cos\left(\frac{n\pi x}{2}\right)dx,$$

$$du = dx, \quad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right).$$

Then,

$$I_2 = (x+1)\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 - \frac{2}{n\pi}\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx.$$

$$(x+1)\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 = -\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right),$$

$$\int_1^2 \sin\left(\frac{n\pi x}{2}\right)dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 = -\frac{2}{n\pi}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right].$$

Therefore,

$$I_2 = -\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right].$$

You can simplify this using properties of sin and cos
Therefiore,

$$a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right],$$

or,

$$a_n = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right].$$

Fourier Sine Series

The Fourier sine series is

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right),$$

where

$$b_n = \int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (x+1)\sin\left(\frac{n\pi x}{2}\right)dx = J_1 + J_2.$$

$$J_1 = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 = -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right].$$

Using integral by parts for J_2:

$$u = x+1, \quad dv = \sin\left(\frac{n\pi x}{2}\right)dx,$$

$$du = dx, \quad v = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right).$$

Then,

$$J_2 = -(x+1)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 + \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx.$$

Boundary term:

$$-(x+1)\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2 = -\frac{6}{n\pi}(-1)^n + \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right).$$

$$\int_1^2 \cos\left(\frac{n\pi x}{2}\right) dx = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_1^2 = -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

Therefore,

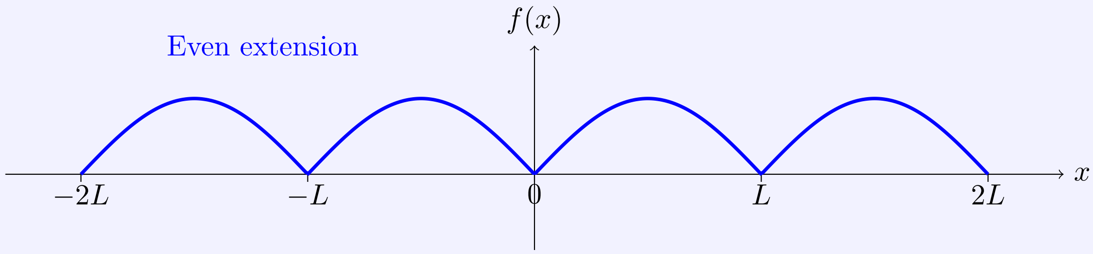
$$J_2 = \frac{6}{n\pi}(-1)^{n+1} + \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right).$$

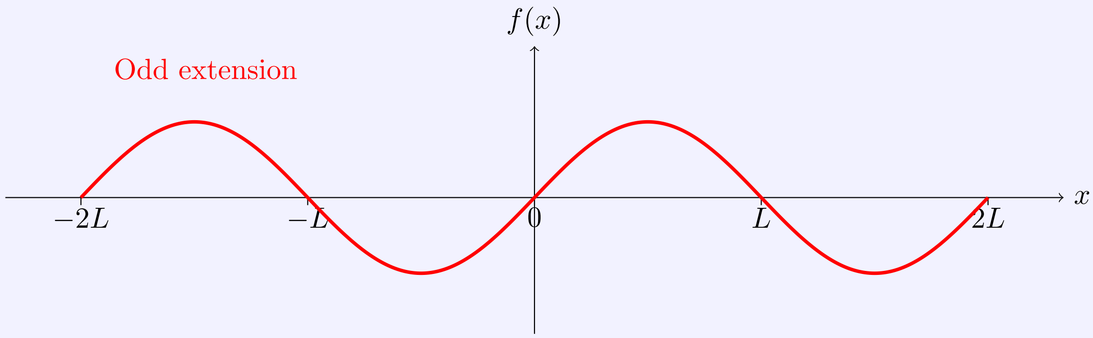
$$b_n = -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right] + \frac{6}{n\pi}(-1)^{n+1} + \frac{4}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right).$$

Verify all the above calculations

(c) $f(x) = \sin(\pi x/L), \quad 0 \le x \le L$

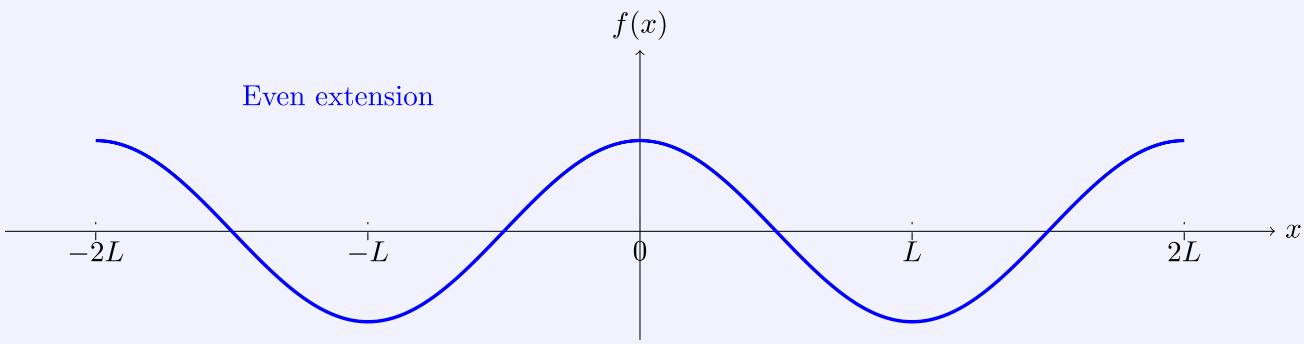
Solution

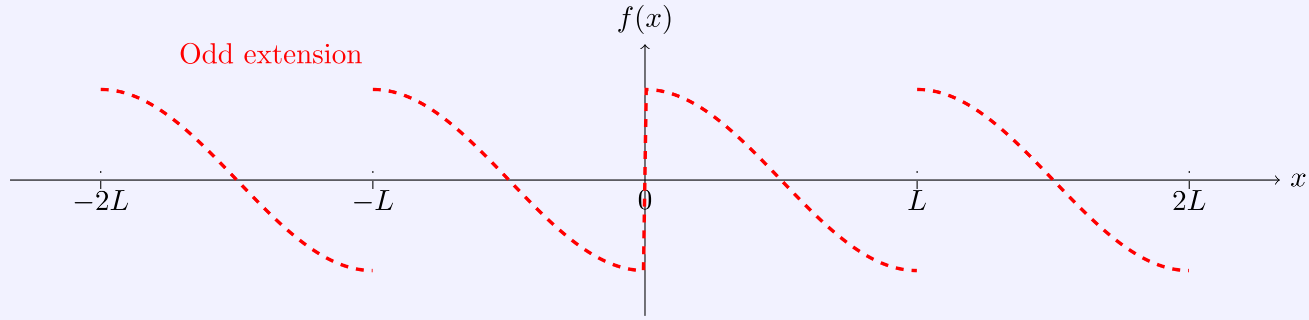




(d) $f(x) = \cos(\pi x/L), \quad 0 \le x \le L$

Solution





Problem 3 (Do not submit) Heat equation - Apply the method of separation of variables to solve the heat equation

$$u_t = 2u_{xx} \text{ for } t > 0, \; 0 \le x \le 4,$$

with boundary conditions

$$u_x(0,t) = u(4,t) = 0,$$

and initial condition

$$u(x,0) = \cos(7\pi x/8).$$

Problem 4 (Do not Submit) Heat equation - Apply the method of separation of variables to solve the heat equation

$$u_t = 2u_{xx} \text{ for } t > 0, \; -1 \le x \le 1,$$

with boundary conditions

$$u(-1,t) = u(1,t), \text{ and } u_x(-1,t) = u_x(1,t)$$

and initial condition

$$u(x,0) = \cos(x) + \sin(x).$$

Problem 5 (Do not submit): Apply the method of separation of variables to determine a solution to the one dimensional heat equation with homogeneous Neumann boundary conditions, i.e.

$$\begin{aligned} &\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \\ &\text{BC:} \quad \frac{\partial u(0,t)}{\partial x} = 0 \text{ and } \frac{\partial u(\pi,t)}{\partial x} = 0 \\ &\text{IC:} \quad u(x,0) = \cos \gamma x, \quad 0 \le x \le \pi \end{aligned}$$

Distinguish between the cases in which γ is and is not an integer. Show by evaluating $u(\pi,0)$ that if γ is not an integer then:

$$\cot \pi\gamma = \frac{1}{\pi}\left[\frac{1}{\gamma} - \sum_{n=1}^{\infty} \frac{2\gamma}{n^2 - \gamma^2}\right]$$

Solution

Assume separable solution

$$u(x,t) = X(x)T(t).$$

Substitute into PDE and separate variables:

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda^2,$$

with $\lambda \neq 0$.
Then

$$T(t) = Ce^{-\alpha^2\lambda^2 t},$$

and

$$X'' + \lambda^2 X = 0,$$

with boundary conditions

$$X'(0) = 0, \quad X'(\pi) = 0.$$

The general solution is

$$X = A\cos\lambda x + B\sin\lambda x.$$

Applying $X'(0) = 0$:

$$X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x,$$

so

$$X'(0) = B\lambda = 0 \implies B = 0.$$

Applying $X'(\pi) = 0$:

$$X'(\pi) = -A\lambda\sin\lambda\pi = 0,$$

which implies

$$\sin\lambda\pi = 0 \implies \lambda = n, \quad n = 1, 2, \ldots$$

Eigenfunctions are

$$X_n = \cos nx.$$

For $\lambda = 0$, the solution is

$$X = Ax + B,$$

and applying $X'(0) = A = 0$ makes the BC at π hold automatically.
Thus the eigenfunction for $\lambda = 0$ is the constant function

$$X_0 = B \cdot 1.$$

By superposition (linearity), the solution is

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) e^{-\alpha^2 n^2 t}.$$

Apply initial condition:

$$\cos\gamma x = A_0 + \sum_{n=1}^{\infty} A_n \cos(nx).$$

Using half-range cosine Fourier expansion,

$$A_0 = \frac{a_0}{2}, \quad A_n = a_n,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos\gamma x\, dx = \begin{cases} \dfrac{2\sin(\gamma\pi)}{\gamma\pi}, & \gamma \notin \mathbb{Z}, \\ 0, & \gamma \in \mathbb{Z}\setminus\{0\}, \\ 2, & \gamma = 0. \end{cases}$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi \cos\gamma x \cos nx \, dx.$$

For $\gamma \neq n$,

$$a_n = \frac{2}{\pi}\int_0^\pi \cos\gamma x \cos nx \, dx = \frac{2}{\pi}\left[\int_0^\pi \cos(\gamma+n)x \, dx + \int_0^\pi \cos(\gamma-n)x \, dx\right]\frac{1}{2},$$

Therefore,

$$a_n = \frac{1}{\pi}\left[\frac{\sin(\gamma+n)\pi}{\gamma+n} + \frac{\sin(\gamma-n)\pi}{\gamma-n}\right].$$

For $\gamma \in \mathbb{Z}$,

$$a_n = \frac{2}{\pi}\int_0^\pi \cos(\gamma x)\cos(nx)\, dx = \begin{cases} 0, & \gamma \neq n, \\ 1, & \gamma = n. \end{cases}$$

For the solution $u(x,t)$, three cases arise depending on γ:

$$u(x,t) = \begin{cases} 1, & \gamma = 0 \in \mathbb{Z}, \\ e^{-\alpha^2 n^2 t}\cos(nx), & \gamma = n \in \mathbb{Z}, \\ \dfrac{\sin(\gamma\pi)}{\gamma\pi} + \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\left\{\frac{\sin((\gamma+n)\pi)}{\gamma+n} + \frac{\sin((\gamma-n)\pi)}{\gamma-n}\right\}\cos(nx)e^{-\alpha^2 n^2 t}, & \gamma \notin \mathbb{Z}. \end{cases}$$

Evaluating $u(\pi, 0) = \cos(\gamma\pi)$, we have:

$$\cos(\gamma\pi) = \frac{\sin(\gamma\pi)}{\gamma\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}(\gamma-n)\frac{\sin(\gamma\pi)\cos(n\pi) + \cos(\gamma\pi)\sin(n\pi)}{\gamma^2-n^2}\cos(n\pi)$$

$$+(\gamma+n)\frac{\sin(\gamma\pi)\cos(n\pi) - \cos(\gamma\pi)\sin(n\pi)}{\gamma^2-n^2}\cos(n\pi).$$

Using trigonometric identities and simplify:

$$= \frac{\sin(\gamma\pi)}{\gamma\pi} + \frac{2\sin(\gamma\pi)}{\pi}\sum_{n=1}^{\infty}\frac{\gamma}{\gamma^2-n^2}(\cos(n\pi))^2.$$

Hence,

$$\boxed{\cos(\gamma\pi) = \frac{1}{\pi}\left(\frac{1}{\gamma} - 2\gamma\sum_{n=1}^{\infty}\frac{1}{n^2-\gamma^2}\right).}$$

Problem 6 (Do not submit): Apply the method of separation of variables to find the temperature in a laterally insulated bar with length L and thermal diffusion coefficient α^2 whose ends are kept at temperature zero and its temperature initially is

$$u(x,0) = f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{L}{2} \\ L-x & \text{if } \frac{L}{2} \le x \le L \end{cases}$$

Problem 7 (Do not submit): Assume that f has a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L), \quad 0 \le x \le L$$

(a) Show formally that

$$\frac{2}{L}\int_0^L [f(x)]^2 = \sum_{n=1}^{\infty} b_n^2.$$

Compare this result (Parseval's equation) with that of **Problem 17 in Section 10.3. in '***Elementary Differential Equations and Boundary Value Problems' by Boyce & DiPrima.***'** What is the corresponding result if f has a cosine series?
(b) Apply the result of part (a) to the series for the sawtooth wave given in **Eq. (9)** of '*Elementary Differential Equations and Boundary Value Problems' by Boyce & DiPrima.*', and thereby show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Problem 8 (Do not submit) Heat equation - Apply the method of separation of variables to solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 2, \quad t > 0$$

$$\text{IC:} \quad u(x,0) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x \le 2 \end{cases}$$

subject to the following boundary conditions. For each case sketch the extension of the initial condition that you would assume on the interval $-6 \le x \le 6$.

(a) $\frac{\partial u(0,t)}{\partial x} = 0$ and $\frac{\partial u(2,t)}{\partial x} = 0$

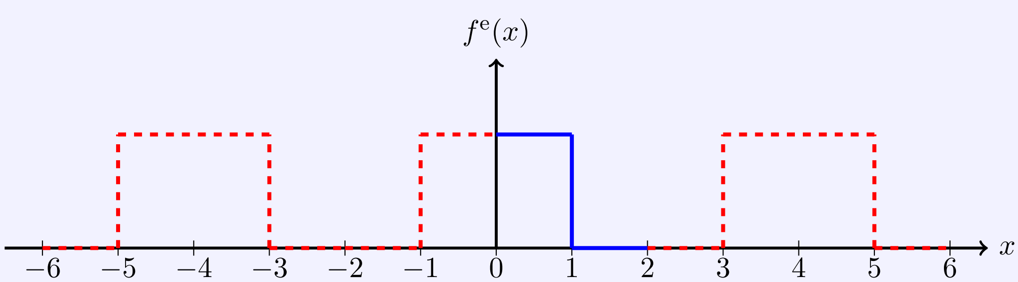
Solution

$\frac{\partial u}{\partial x}(0,t) = 0, \frac{\partial u}{\partial x}(2,t) = 0$ Neumann B.C.'s Separation of variables gives:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,0) = f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Assume an even extension of $f(x)$:



$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cdot\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{2}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx$$
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1$$
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$
$$a_0 = \frac{1}{2}\cdot 2\int_0^1 1\cdot dx = 1$$
$$f(x) = u(x,0) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi x}{2}\right)$$
$$\boxed{u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi x}{2}\right)\cdot e^{-\left(\frac{n\pi}{2}\right)^2 t}}$$

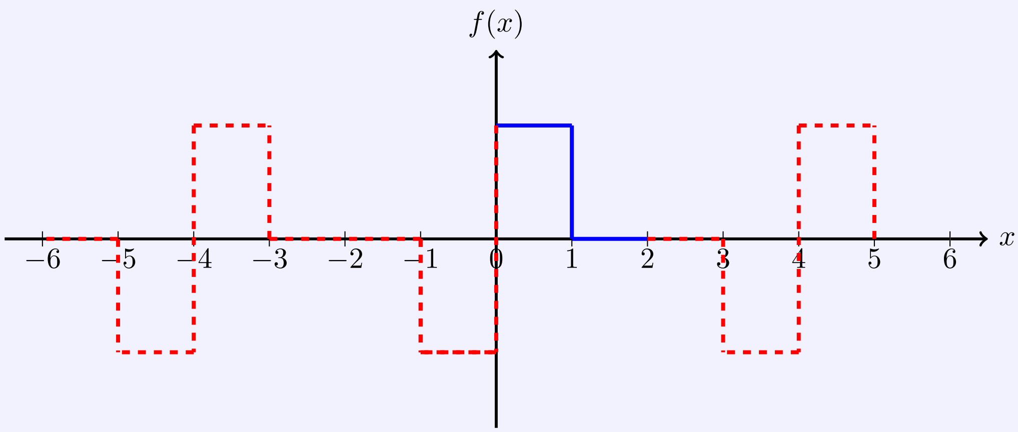
(b) $u(0,t) = 0$ and $u(2,t) = 0$

Solution

$u(0,t) = 0, u(2,t) = 0$ Dirichlet B.C.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2 t}$$
$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Assume an odd extension of $f(x)$:



$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f^{\text{odd}}(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)dx \\
&= \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 \\
&= \frac{-2}{n\pi}\left[\text{Cos}\left(\frac{n\pi}{2}\right)-1\right] \\
u(x,0) &= f(x) = \sum_{n=1}^{\infty}\frac{-2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-1\right]\sin\left(\frac{n\pi x}{2}\right)
\end{aligned}
$$

$$
\boxed{u(x,t) = \sum_{n=1}^{n=1} -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-1\right]\sin\left(\frac{n\pi x}{2}\right)e^{-\left(\frac{n\pi}{2}\right)^2 t}}
$$

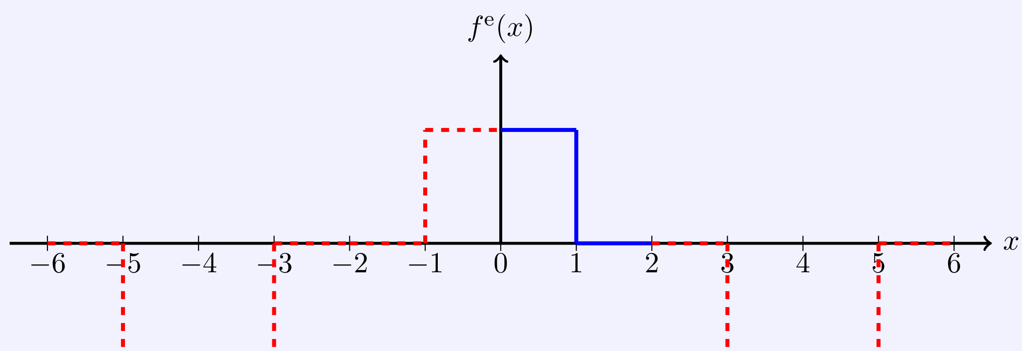
(c) $\frac{\partial u(0,t)}{\partial x} = 0$ and $u(2,t) = 0$

Solution

$\frac{\partial u}{\partial x}(0,t) = 0$, and $u(2,t) = 0$ Mixed type B.C.B

$$
\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} a_n \cos\left(\frac{(2n+1)kx}{2L}\right)e^{-\left(\frac{(2n+1)\pi}{2L}\right)^2 t} \\
u(x,0) &= \sum_{n=0}^{\infty} a_n \cos\left(\frac{(2n+1)\pi x}{2L}\right)
\end{aligned}
$$

extension of $f(x)$, where $2L = 4 \implies L = 2$:



Compare this to the fundamental mode, Cos $\left(\frac{\pi x}{4}\right)$:

$$
\begin{aligned}
a_n = \frac{2}{2}\int_0^1 \cos\left(\frac{(2n+1)\pi x}{4}\right)\cdot dx &= \frac{4}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi x}{4}\right)\Big|_0^1 \\
&= \frac{4}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi x}{4}\right)\Big|_0^1 \\
&= \frac{4}{(3n+1)\pi}\sin\left(\frac{(2n+1)\pi}{4}\right)
\end{aligned}
$$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \cdot \sin\left(\frac{(2n+1)\pi}{4}\right) \cdot \cos\left(\frac{(2n+1)\pi x}{4}\right)$$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} \frac{4\sin\left(\frac{(2n+1)\pi}{2}\right)}{(2n+1)\pi} \cos\left(\frac{(2n+1)\pi x}{4}\right) e^{-\left(\frac{(2n+1)\pi}{4}\right)^2 t}}$$

(d) $u(0,t) = 0$ and $\frac{\partial u(2,t)}{\partial x} = 0$

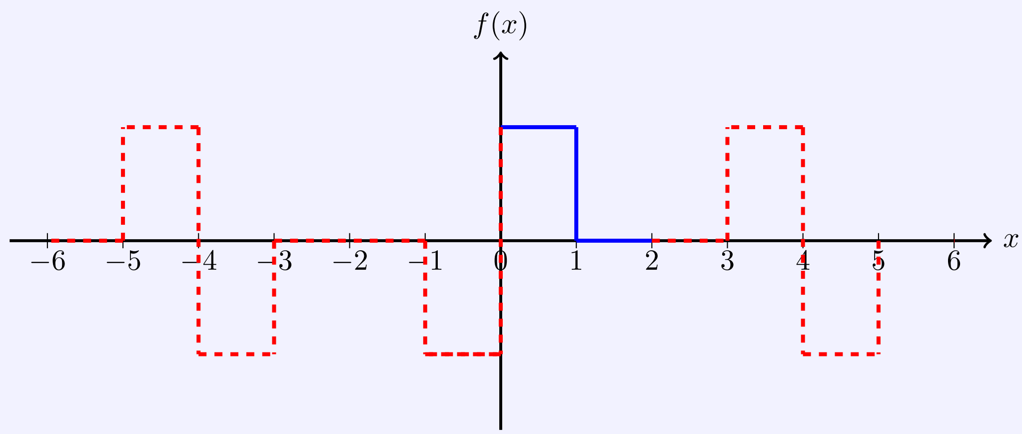
Solution

$u(0,t) = 0$ and $\frac{\partial u}{\partial x}(2,t) = 0$, Mixed type B.C. A

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi(x}{2L}\right) e^{-\left(\frac{(2n+1)\pi}{2L}\right)^2 t}$$

$$u(x,0) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

extension of $f(x)$:



Compare to the fundamental mode $\sin\left(\frac{\pi x}{4}\right)$

$$\begin{aligned} b_n &= \frac{2}{2}\int_0^1 \sin\left(\frac{(2n+1)\pi x}{4}\right) dx \\ &= \frac{-4}{(2n+1)\pi} \cos\left(\frac{(2n+1)\pi x}{4}\right)\Bigg|_0^1 \\ &= \frac{-4}{(2n+1)\pi}\left[\cos\left(\frac{(2n+1)\pi}{4}\right) - 1\right] \end{aligned}$$

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} \frac{-4}{(2n+1)\pi}\left[\cos\left(\frac{(2n+1)\pi}{4}\right) - 1\right] \sin\left(\frac{(2n+1)\pi x}{4}\right)$$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} b_n \cdot \sin\left(\frac{(2n+1)\pi x}{4}\right) \cdot e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t}}$$

Problem 9 (Do not submit) Complex Fourier series - Find the complex Fourier series of $f(x) = e^{\alpha x}$ specified on $-1 \le x \le 1$, assuming a period of $2L = 2$.

Solution

The complex Fourier series for a function with period 2 is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x},$$

where

$$c_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx.$$

Substitute $f(x) = e^{\alpha x}$:

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{\alpha x}e^{-in\pi x}dx = \frac{1}{2}\int_{-1}^{1} e^{(\alpha - in\pi)x}dx.$$

This gives:

$$c_n = \frac{1}{2}\left.\frac{e^{(\alpha-in\pi)x}}{\alpha - in\pi}\right|_{x=-1}^{x=1} = \frac{1}{2}\cdot\frac{e^{\alpha-in\pi} - e^{-\alpha+in\pi}}{\alpha - in\pi}.$$

Simplify
Recall that $e^{-in\pi} = \cos(n\pi) - i\sin(n\pi) = (-1)^n$, so

$$e^{\alpha-in\pi} = e^{\alpha}(-1)^n, \quad e^{-\alpha+in\pi} = e^{-\alpha}(-1)^n.$$

Thus,

$$c_n = \frac{1}{2}\cdot\frac{(-1)^n(e^{\alpha} - e^{-\alpha})}{\alpha - in\pi} = (-1)^n\frac{\sinh\alpha}{\alpha - in\pi}.$$

Therefore,

$$\boxed{f(x) = e^{\alpha x} = \sum_{n=-\infty}^{\infty}(-1)^n\frac{\sinh\alpha}{\alpha - in\pi}e^{in\pi x}.}$$

Problem 10 (Do not submit) Heat equation with inhomogeneous boundary conditions - Consider the following boundary value problem for the heat equation governing the temperature within a conducting bar:

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0 \\ BC &: u_x(0,t) = 1, \quad u(1,t) = 0 \\ IC &: u(x,0) = 0 \end{aligned}$$

(a) Determine the steady-state temperature $u_\infty(x)$.
(b) Let $u(x,t) = u_\infty(x) + v(x,t)$ and identify the PDE, BC and IC satisfied by $v(x,t)$.
(c) Use the method of separation of variables to solve the above boundary value problem for $v(x,t)$ and from this determine the solution $u(x,t)$.

Solution

(a) Steady-state solution $u_\infty(x)$
At steady state, $u_t = 0$, so

$$u_\infty''(x) = 0.$$

Integrate twice to get:

$$u_\infty(x) = Ax + B,$$

where A, B are constants.
Apply boundary conditions:

$$u'_\infty(0) = A = 1,$$

$$u_\infty(1) = A \cdot 1 + B = 1 + B = 0 \implies B = -1.$$

Thus,

$$\boxed{u_\infty(x) = x - 1.}$$

(b) Find PDE, BCs, IC for v Let

$$u = u_\infty + v,$$

substitute into PDE:

$$u_t = u_{xx} \implies v_t = v_{xx} + u''_\infty(x) = v_{xx} + 0 = v_{xx},$$

Boundary conditions:

$$u_x(0,t) = u'_\infty(0) + v_x(0,t) = 1 \implies 1 + v_x(0,t) = 1 \implies v_x(0,t) = 0,$$

$$u(1,t) = u_\infty(1) + v(1,t) = 0 \implies 0 + v(1,t) = 0 \implies v(1,t) = 0.$$

Initial condition:

$$v(x,0) = u(x,0) - u_\infty(x) = 0 - (x-1) = 1 - x.$$

Therefore,

$$\boxed{\begin{cases} v_t = v_{xx}, & 0 < x < 1, \quad t > 0, \\ v_x(0,t) = 0, & v(1,t) = 0, \\ v(x,0) = 1 - x, & 0 \le x \le 1. \end{cases}}$$

(c) Solve for $v(x,t)$ using separation of variables
Consider

$$v_t = v_{xx}, \quad 0 < x < 1,$$

with mixed boundary conditions:

$$v_x(0,t) = 0, \quad v(1,t) = 0.$$

We let

$$v(x,t) = X(x)T(t).$$

Substitute into PDE:

$$X(x)T'(t) = X''(x)T(t) \implies \frac{T'}{T} = \frac{X''}{X} = -\lambda,$$

ODEs:

$$T'(t) + \lambda T(t) = 0, \quad \Rightarrow \quad T(t) = e^{-\lambda t},$$

$$X''(x) + \lambda X(x) = 0,$$

with BCs:

$$X'(0) = 0, \quad X(1) = 0.$$

General solution:

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Apply BC at $x = 0$:

$$X'(0) = -A\sqrt{\lambda}\sin(0) + B\sqrt{\lambda}\cos(0) = B\sqrt{\lambda} = 0 \implies B = 0.$$

Apply BC at $x = 1$:

$$X(1) = A\cos(\sqrt{\lambda}\cdot 1) = 0.$$

For nontrivial $A \neq 0$, hence we require

$$\cos(\sqrt{\lambda}) = 0 \implies \sqrt{\lambda} = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, 3, \dots$$

Hence eigenvalues are:

$$\lambda_n = \left(\frac{(2n+1)\pi}{2}\right)^2.$$

Corresponding eigenfunctions:

$$X_n(x) = \cos\left(\frac{(2n+1)\pi}{2}x\right).$$

The general solution for $v(x,t)$

$$v(x,t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} X_n(x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t} \cos\left(\frac{(2n+1)\pi}{2}x\right).$$

Using initial condition

$$v(x,0) = 1 - x = \sum_{n=1}^{\infty} c_n \cos\left(\frac{(2n+1)\pi}{2}x\right).$$

using orthogonality:

$$c_m = 2\int_0^1 (1+x)\cos\left(\frac{(2m-1)\pi}{2}x\right)dx.$$

$$c_m = 2\int_0^1 \cos(\alpha_m x)dx - 2\int_0^1 x\cos(\alpha_m x)dx,$$

where

$$\alpha_m = \frac{(2m+1)\pi}{2}.$$

$$\int_0^1 \cos(\alpha_m x)dx = \left.\frac{\sin(\alpha_m x)}{\alpha_m}\right|_0^1 = \frac{\sin(\alpha_m)}{\alpha_m}.$$

Use integration by parts:

$$\int_0^1 x\cos(\alpha_m x)dx = \left. x\frac{\sin(\alpha_m x)}{\alpha_m}\right|_0^1 - \int_0^1 \frac{\sin(\alpha_m x)}{\alpha_m}dx = \frac{\sin(\alpha_m)}{\alpha_m} - \frac{1}{\alpha_m}\int_0^1 \sin(\alpha_m x)dx.$$

$$\int_0^1 \sin(\alpha_m x)dx = -\left.\frac{\cos(\alpha_m x)}{\alpha_m}\right|_0^1 = \frac{1-\cos(\alpha_m)}{\alpha_m}.$$

Therefore,

$$\int_0^1 x\cos(\alpha_m x)dx = \frac{\sin(\alpha_m)}{\alpha_m} - \frac{1-\cos(\alpha_m)}{\alpha_m^2}.$$

or:

$$c_m = 2\cdot\frac{\sin(\alpha_m)}{\alpha_m} - 2\left(\frac{\sin(\alpha_m)}{\alpha_m} - \frac{1-\cos(\alpha_m)}{\alpha_m^2}\right) = \cdot\frac{2}{\alpha_m^2}.$$

Therefore

$$\boxed{u(x,t) = u_\infty(x) + v(x,t) = x - 1 + \sum_{n=0}^{\infty} 2\frac{8}{(2n+1)^2} e^{-\left(\frac{(2n+1)\pi}{2}\right)^2 t} \cos\left(\frac{(2n+1)\pi}{2}x\right).}$$

Problem 11 (Do not submit) Heat equation with inhomogeneous boundary conditions - Solve the following heat conduction problem with heat loss and a distributed heat source

$$u_t = u_{xx} - u + x, \quad 0 < x < 1,\ t > 0$$
$$\text{BC: } u_x(0,t) = 1, \quad u_x(1,t) = 2$$
$$\text{IC: } u(x,0) = x$$

Hint: First try to find a steady-state solution that satisfies the PDE and the inhmogeneous boundary conditions. Next, let $u(x,t) = u_\infty(x) + v(x,t)$ and identify the PDE, BC and IC satisfied by $v(x,t)$.

It may be helpful to know:

$$\int \cos(Ax)\cosh(Bx)dx = \frac{B\cos(Ax)\sinh(Bx) + A\sin(Ax)\cosh(Bx)}{A^2 + B^2}$$

Solution

Consider the steady-state problem

$$u''_\infty(x) - u_\infty(x) = -x,$$

with boundary conditions

$$u'_\infty(0) = 1, \quad u'_\infty(1) = 2.$$

The general solution of the homogeneous equation $u'' - u = 0$ is

$$u_h(x) = C_1 \cosh x + C_2 \sinh x.$$

A particular solution can be found by variation of parameters or undetermined coefficients:

$$u_p(x) = x,$$

since

$$u''_p - u_p = 0 - x = -x,$$

Thus,

$$u_\infty(x) = C_1 \cosh x + C_2 \sinh x + x.$$

Applying the boundary conditions:

$$u'_\infty(x) = C_1 \sinh x + C_2 \cosh x + 1.$$

At $x = 0$:

$$u'_\infty(0) = C_1 \cdot 0 + C_2 \cdot 1 + 1 = C_2 + 1 = 1 \implies C_2 = 0.$$

At $x = 1$:

$$u'_\infty(1) = C_1 \sinh 1 + 0 + 1 = 2 \implies C_1 = \frac{1}{\sinh 1}.$$

Therefore, the steady-state solution is

$$\boxed{u_\infty(x) = \frac{\cosh x}{\sinh 1} + x.}$$

Define $v(x,t) = u(x,t) - u_\infty(x)$. Then v satisfies

$$v_t = v_{xx} - v,$$

with homogeneous Neumann boundary conditions

$$v_x(0,t) = 0, \quad v_x(1,t) = 0,$$

and initial condition

$$v(x,0) = u(x,0) - u_\infty(x) = x - \left(\frac{\cosh x}{\sinh 1} + x\right) = -\frac{\cosh x}{\sinh 1}.$$

Show that separation of variables on the PDE for v gives Fourier cosine series:

$$v(x,t) = \sum_{n=0}^{\infty} a_n e^{-(\mu_n^2+1)t}\cos(\mu_n x),$$

where $\mu_n = n\pi$, and the coefficients are **(Derive the expressions)**

$$a_0 = \int_0^1 v(x,0)dx = -2, \quad a_n = 2\int_0^1 v(x,0)\cos(n\pi x)dx = \frac{2(-1)^{n+1}}{(n\pi)^2+1}, \quad n \geq 1.$$

Use the given formula to evaluate the integrals

$$\boxed{u(x,t) = \frac{\cosh x}{\sinh 1} + x + \frac{a_0 e^{-t}}{2} + \sum_{n=1}^{\infty} a_n e^{-(n^2\pi^2+1)t}\cos(n\pi x).}$$

Problem 12: (Do not submit) Steady state solutions - Find the steady-state solutions for the following heat conduction boundary value problems:

a) $u_t = \alpha^2 u_{xx}$, $u(0,t) = 1$, $u(\pi,t) = 2$

b) $u_t = \alpha^2 u_{xx}$, $u(0,t) = 5$, $u_x(1,t) = 0$

c) $u_t = \alpha^2 u_{xx}$, $u(0,t) = 0$, $u_x(2,t) + u(2,t) = 4$

d) $u_t = u_{xx} - \beta^2 u$, $u(0,t) = 1$, $u(\pi,t) = 2$

e) $u_t = u_{xx} - \beta^2 u$, $u_x(0,t) = 1$, $u(\pi,t) = 2$

Solution

(a)

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = 1, \quad u(\pi,t) = 2.$$

Steady-state: $u_t = 0 \implies u_{xx} = 0$.
General solution:

$$u_\infty(x) = Ax + B.$$

Apply BCs:

$$u_\infty(0) = B = 1,$$

$$u_\infty(\pi) = A\pi + B = 2 \implies A = \frac{2-1}{\pi} = \frac{1}{\pi}.$$

$$\boxed{u_\infty(x) = \frac{x}{\pi} + 1.}$$

(b)

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = 5, \quad u_x(1,t) = 0.$$

Steady-state:

$$u_{xx} = 0 \implies u_\infty(x) = Ax + B.$$

BCs:

$$u_\infty(0) = B = 5,$$

$$u'_\infty(x) = A, \quad u'_\infty(1) = A = 0 \implies A = 0.$$

Hence,

$$\boxed{u_\infty(x) = 5.}$$

(c)

$$u_t = \alpha^2 u_{xx}, \quad u(0,t) = 0, \quad u_x(2,t) + u(2,t) = 4.$$

Steady-state:

$$u_{xx} = 0 \implies u_\infty(x) = Ax + B.$$

BCs:

$$u_\infty(0) = B = 0,$$

$$u'_\infty(x) = A, \quad u'_\infty(2) + u_\infty(2) = A + (2A + B) = A + 2A + 0 = 3A = 4 \implies A = \frac{4}{3}.$$

Therefore,

$$\boxed{u_\infty(x) = \frac{4}{3}x.}$$

(d)

$$u_t = u_{xx} - \beta^2 u, \quad u(0,t) = 1, \quad u(\pi,t) = 2.$$

Steady-state:

$$u_{xx} - \beta^2 u = 0.$$

General solution:

$$u_\infty(x) = C_1 \cosh(\beta x) + C_2 \sinh(\beta x).$$

Apply boundary conditions:

$$u_\infty(0) = C_1 = 1,$$

$$u_\infty(\pi) = C_1 \cosh(\beta\pi) + C_2 \sinh(\beta\pi) = 2.$$

Solve for C_2:

$$1 \cdot \cosh(\beta\pi) + C_2 \sinh(\beta\pi) = 2 \implies C_2 = \frac{2 - \cosh(\beta\pi)}{\sinh(\beta\pi)}.$$

Therefore, the steady-state solution is

$$\boxed{u_\infty(x) = \cosh(\beta x) + \frac{2 - \cosh(\beta\pi)}{\sinh(\beta\pi)} \sinh(\beta x).}$$

(e)

$$u_t = u_{xx} - \beta^2 u, \quad u_x(0,t) = 1, \quad u(\pi,t) = 2.$$

Steady-state:

$$u_{xx} - \beta^2 u = 0.$$

General solution:

$$u_\infty(x) = C_1 \cosh(\beta x) + C_2 \sinh(\beta x).$$

$$u'_\infty(x) = \beta C_1 \sinh(\beta x) + \beta C_2 \cosh(\beta x).$$

Apply BC at $x = 0$:

$$u'_\infty(0) = \beta C_2 = 1 \implies C_2 = \frac{1}{\beta}.$$

Apply BC at $x = \pi$:

$$u_\infty(\pi) = C_1 \cosh(\beta\pi) + C_2 \sinh(\beta\pi) = 2.$$

Substitute C_2:

$$C_1\cosh(\beta\pi)+\frac{1}{\beta}\sinh(\beta\pi)=2 \implies C_1=\frac{2-\frac{1}{\beta}\sinh(\beta\pi)}{\cosh(\beta\pi)}.$$

Thus the steady-state solution is

$$\boxed{u_\infty(x)=\frac{2-\frac{1}{\beta}\sinh(\beta\pi)}{\cosh(\beta\pi)}\cosh(\beta x)+\frac{1}{\beta}\sinh(\beta x).}$$

Problem 13 (Do not submit): Solve the following heat conduction problem with a distributed heat source

$$\begin{aligned} u_t &= 16u_{xx}+\cos\left(\frac{7\pi x}{4}\right), \quad 0<x<2,\ t>0 \\ \text{BC: } & u_x(0,t)=1=u_x(2,t) \\ \text{IC: } & u(x,0)=x^2-4 \end{aligned}$$

Problem 14 (Submit only the final solution): Solve the following heat conduction problem with heat loss and a distributed heat source

$$\begin{aligned} u_t &= u_{xx}-u+x, \quad 0<x<1,\ t>0 \\ \text{BC: } & u_x(0,t)=1, \quad u_x(1,t)=2 \\ \text{IC: } & u(x,0)=x \end{aligned}$$

For Problem 14, submit only the final solution, not the workout
Hint: First try to find a steady-state solution that satisfies the PDE and the inhomogeneous boundary conditions.

Solution

See problem 11

Problem 15 (Do not submit): Solve the initial boundary value problem:

$$\begin{aligned} u_t &= u_{xx}+e^{-3t}\cos\left(\frac{7\pi}{2}x\right)+1, \quad 0<x<1,\ t>0 \\ \text{BC: } & u_x(0,t)=1, \quad u(1,t)=t \\ \text{IC: } & u(x,0)=1 \end{aligned}$$

Problem 16 (Submit): Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$\begin{aligned} u_t &= \alpha^2 u_{xx}+1-xe^{-t},\ 0<x<1,\ t>0 \\ u(0,t) &= e^{-t}, \text{ and } u_x(1,t)=t \\ u(x,0) &= \sin\left(\frac{\pi x}{2}\right)+1. \end{aligned}$$

Solution

Find a particular solution satisfying the boundary conditions
Let $U_p(x,t)=A(t)x+B(t)$

Boundary conditions for U_p:

$$U_p(0,t) = B(t) = e^{-t},$$

$$U_p'(1,t) = A(t) = t.$$

So,

$$\boxed{U_p(x,t) = tx + e^{-t}.}$$

Write $u = U_p + v$, where $v(x,t)$ satisfies homogeneous boundary conditions

BCs for v:

$$v(0,t) = u(0,t) - U_p(0,t) = e^{-t} - e^{-t} = 0,$$

$$v_x(1,t) = u_x(1,t) - U_p'(1,t) = t - t = 0.$$

Initial condition for v:

$$v(x,0) = u(x,0) - U_p(x,0) = \sin\left(\frac{\pi x}{2}\right) + 1 - \left(0 \cdot x + e^0\right) = \sin\left(\frac{\pi x}{2}\right).$$

PDE for v

Since

$$u_t = v_t + U_p'(t)x - e^{-t},$$

and

$$u_{xx} = v_{xx} + 0,$$

substitute into PDE:

$$u_t = \alpha^2 u_{xx} + 1 - xe^{-t} \implies v_t + A'(t)x + B'(t) = \alpha^2 v_{xx} + 1 - xe^{-t}.$$

Recall $A(t) = t$, $B(t) = e^{-t}$, so

$$A'(t) = 1, \quad B'(t) = -e^{-t}.$$

Simplify to get

$$\boxed{v_t = \alpha^2 v_{xx} + (1 + e^{-t})(1 - x), \quad 0 < x < 1, \quad t > 0,}$$

with homogeneous BCs:

$$v(0,t) = 0, \quad v_x(1,t) = 0,$$

and initial condition

$$v(x,0) = \sin\left(\frac{\pi x}{2}\right).$$

Solve PDE for v by eigenfunction expansion

Eigenvalues are:

$$\lambda_n = \mu_n^2, \quad \mu_n = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, 3, \ldots,$$

and eigenfunctions

$$X_n(x) = \sin(\mu_n x).$$

Expand

$$v(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x).$$

Also expand the source term

$$f(x,t) = (1 + e^{-t})(1 - x) = \sum_{n=0}^{\infty} f_n(t) X_n(x),$$

where

$$f_n(t) = 2(1+e^{-t}) \int_0^1 (1-x)\sin(\mu_n x)dx = S_n(1+e^{-t}).$$

with

$$S_n = \frac{1}{\mu_n} - \frac{(-1)^n}{\mu_n^2}$$

Solve ODEs for $T_n(t)$
Substitute back the expansions into the PDE

$$\sum_{n=0}^{\infty} \left[T_n'(t) + \alpha^2\mu_n^2 T_n(t) - f_n(t)\right]\sin(\mu_n x) = 0$$

Each coefficient vanishes, hence

$$T_n'(t) + \alpha^2\mu_n^2 T_n(t) = f_n(t),$$

Solve the ODE by integrating factor:

$$T_n(t) = e^{-\alpha^2\mu_n^2 t}\left[c_n + \int_0^t e^{\alpha^2\mu_n^2 s} f_n(s)ds\right].$$

Simplify to get

$$T_n(t) = S_n\left[\frac{1}{(\alpha\mu_n)^2} + \frac{e^{-t}}{(\alpha\mu_n)^2 - 1}\right] + c_n e^{-\alpha^2\mu_n^2 t}$$

Therefore we get

$$v(x,t) = \sum_{n=0}^{\infty}\left\{S_n\left[\frac{1}{(\alpha\mu_n)^2} + \frac{e^{-t}}{(\alpha\mu_n)^2 - 1}\right] + c_n e^{-\alpha^2\mu_n^2 t}\right\} X_n(x).$$

Use IC to get c_n

$$v(x,0) = \sum_{n=0}^{\infty}\left\{S_n\left[\frac{1}{(\alpha\mu_n)^2} + \frac{1}{(\alpha\mu_n)^2 - 1}\right] + c_n\right\} X_n(x) = \sin\left(\frac{\pi x}{2}\right).$$

Thus

$$\left\{S_n\left[\frac{1}{(\alpha\mu_n)^2} + \frac{1}{(\alpha\mu_n)^2 - 1}\right] + c_n\right\} = \delta_{n0}$$

Final solution

$$\boxed{u(x,t) = U_p(x,t) + \sum_{n=1}^{\infty} T_n(t)\sin(\mu_n x),}$$

with

$$U_p(x,t) = tx + e^{-t},$$

and $T_n(t)$ as above.

Problem 17 (Do not submit): Solve the inhomogeneous heat conduction problem with heat loss, a time dependent source, and subject to time dependent boundary conditions:

$$u_t = u_{xx} - u + e^{-t}\sin(x),\ 0 < x < \frac{\pi}{2},\ t > 0$$
$$u(0,t) = 0,\ \text{and}\ \frac{\partial u(\pi/2,t)}{\partial x} = e^{-t}$$
$$u(x,0) = x.$$

Solution

Let $w(x,t) = A(t)x + B(t)$ such that it matches the BC.

$$0 = W(0,t) = B(t) \quad W_x = A(t) \quad W_x(\pi/2,t) = A(t) = e^{-t} \quad W = xe^{-t} \quad W_t = -xe^{-t}$$

Now let $u(x,t) = w(x,t) + k(x,t)$ and substitute into the PDE
Let $u(x,t) = w(x,t) + v(x,t)$. Then $v(x,t)$ satisfies

$$v_t = v_{xx} - v + e^{-t}\sin x$$
$$v(0,t) = 0,\ v_x\left(\tfrac{\pi}{2},t\right) = 0,$$
$$v(x,0) = u(x,0) - w(x,0) = x - x = 0.$$

Expand $v(x,t)$ in eigenfunctions of the Laplacian with $v(0,t) = 0$ and $v_x(\frac{\pi}{2},t) = 0$, i.e., $\mu_n = 2n+1$ and $X_n(x) = \sin((2n+1)x)$ for $n = 0,1,2,\ldots$

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t)\sin((2n+1)x).$$

Expand also the source term

$$e^{-t}\sin x = \sum_{n=0}^{\infty} s_n(t)\sin(2n+1)x. \Rightarrow s_n(t) = \delta_{n0}e^{-t}$$

Now let

$$v(x,t) = \sum_{n=0}^{\infty} v_n(t)\sin\mu_n x \quad v_t = \sum_{n=0}^{\infty} v_n \sin\mu_n x \quad v_{xx} = \sum_{n=0}^{\infty} v_n\left\{-\mu_n^2\sin\mu_n x\right\}$$

$$\therefore 0 = v_t - v_{xx} + v - e^{-t}\sin x = \sum_{n=0}^{\infty}\left\{\frac{dv_n}{dt} + \left(\mu_n^2+1\right)v_n - e^{-t}\delta_{n_0}\right\}\sin\mu_n x$$

We have $\frac{dv_n}{dt} + \left(1+\mu_n^2\right)v_n = e^{-t}\delta_{n_0} \Rightarrow \frac{d}{dt}\left\{e^{+\left(1+\mu_n^2\right)t}v_n\right\} = e^{\mu_n^2 t}\delta_{n_0}$

$$\therefore \quad e^{\left(1+\mu_n^2\right)_t^1 t}v_n = \int_0^t e^{\mu_n^2\delta_{n_0}d\tau} + c_n = \left(\frac{e^{\mu_n^2 t}-1}{\mu_n^2}\right)\delta_{n0} + c_n.$$

$$\therefore v_n(t) = \left(\frac{e^{-t} - e^{-\left(1+\mu_n^2\right)t}}{\mu_n^2}\right)\delta_{n0} + c_n e^{-\left(1+\mu_n^2\right)t}; v_n(0) = c_n$$

Now

$$0 = \sum_{n=0}^{\infty} V_n(0)\sin\left(\mu_n x\right) \Rightarrow V_n(0) = c_n = 0$$

.

$$\therefore u(x,t) = xe^{-t} + \sum_{n=0}^{\infty}\left(\frac{e^{-t} - e^{-\left(1+\mu_n^2\right)t}}{\mu_n^2}\right)\delta_{n_0}\sin\mu_n x \quad \mu_0 = 1$$
$$= xe^{-t} + \left(\frac{e^{-t} - e^{-2t}}{1}\right)\sin x$$

Problem 18 (Do not submit): Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$
\begin{aligned}
u_t &= u_{xx} + xt + 1,\ 0 < x < 1,\ t > 0 \\
u_x(0,t) &= 0,\ \text{and}\ u(1,t) = t \\
u(x,0) &= 0.
\end{aligned}
$$

Problem 19 (Do not suubmit): The motion of a string on an elastic foundation with a stiffness γ satisfies the following initial-boundary value problem:

$$
\begin{aligned}
u_t &= u_{xx} - u + 1, \quad 0 < x < 1,\ t > 0 \\
\text{BC: } & u(0,t) = 1, \quad u(1,t) = 2 \\
\text{IC: } & u(x,0) = 1
\end{aligned}
$$