Math 257/316 Assignment 5, 2025

Due on Monday June 17 , 2025. Submit online in a PDF document on Canvas by 11:59 pm of the due date

Submit ONLY ONE question by the due date

SET 1

Problem 1: Consider the following initial boundary value problem for the wave equation

$$
\begin{aligned}
&u_{tt} = u_{xx} \qquad t > 0,\ \ 0 \le x \le 1, \\
&\text{BC:} \quad u(0,t) = 0, \ \text{ and } \ u(1,t) = 1 \\
&\text{IC:} \quad u(x,0) = \sin(\pi x) + x, \ \text{ and } \ u_t(x,0) = \sin(3\pi x)
\end{aligned}
$$

(a) After dealing with the inhomogeneous boundary conditions, solve this problem using the method of separation of variables. (*Note:* Only show cases that give non-trivial solutions when solving the eigenvalue problems.)
(b) Find the D'Alembert's solution for the initial boundary value problem that corresponds to the problem stated above with homogeneous boundary conditions. Compare this to the solution you find in part (a).

Problem 2: Solve the following inhomogeneous initial boundary value problem for the wave equation:

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} + e^{-t}\sin(5x),\ \ 0 < x < \frac{\pi}{2},\ \ t > 0 \\
u(0,t) &= 0 \text{ and } u_x(\frac{\pi}{2}, t) = t, \quad t > 0 \\
u(x,0) &= 0, \qquad u_t(x,0) = \sin(3x) + x, \quad 0 < x < \frac{\pi}{2}
\end{aligned}
$$

Problem 3: Consider an infinite string subject to the initial conditions

$$
u(x,0) = f(x) = \begin{cases} x+1 & \text{if } -1 \le x < 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}
$$

$$
u_t(x,0) = g(x) = 0.
$$

Assume that for this string $(T/\rho)^{1/2} = 1$, with T being the tension in the string and ρ the density of the string per its unit length.

In the x-t plane draw the important characteristics that can be interpreted from the D'Alembert's solution. Using these characteristics identify different regions in the x-t plane that have the same expression for the displacement of the string. Find the expression for the displacement in each region. Then, use this geometric interpretation of the D'Alembert's solution to sketch the shape of the string at times $t = 0$, $1/2$, 1 and 2.

Problem 4: The motion of a string subject to a gravitational load satisfies the following initial-boundary value problem:

$$u_{tt} = c^2 u_{xx} - g, \ 0 < x < 1, \ \ t > 0 \qquad (1)$$

$$u(0,t) = u(1,t) \ = \ 0 \qquad (2)$$

$$u(x,0) = \sin(\pi x), \ u_t(x,0) \ = \ 0.$$

Here g is the acceleration due to gravity, which you may assume is constant.
(a) Determine the static deflection of the string, which is determined by solving (1) in which it is assumed that $u_{tt} = 0$ subject to the boundary conditions (2).
(b) Use the solution obtained in (a) to reduce the initial-boundary value problem to solving a homogeneous wave equation subject to homogeneous boundary conditions. Now use separation of variables to determine the solution to this boundary value problem and hence the complete solution of the entire initial-boundary value problem.

HINT: The following integral may be useful:

$$\int_0^1 \left(x^2 - x\right) \sin\left(n\pi x\right) dx = 2\frac{\cos n\pi - 1}{n^3\pi^3}$$

SET 2

Problem 1: Consider the static deflection of a plate that satisfies the Laplace's equation and is subject to the following boundary conditions

$$u_{xx} + u_{yy} = 0 \ \text{ for } \ 0 \ \ \leq x \leq 3 \ \text{ and } \ 0 \leq y \leq 2$$

$$u_x(3,y) = 0, \quad u_y(x,0) = 0$$

$$u(x,2) = f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x < 3 \end{cases}$$

$$u(0,y) = f(y) = \begin{cases} y & \text{if } 0 \leq y < 1 \\ 2 - y & \text{if } 1 \leq y < 2 \end{cases}$$

The above boundary conditions state that the plate has zero slope on its $x = 3$ and $y = 0$ sides and undergoes triangular deformations on the $x = 0$ and $y = 2$ sides.

(a) Use the method of separation of variables to solve this boundary value problem.
(b) Verify that the solution you found satisfies all the boundary conditions.

Problem 2: A metal plate occupies a quarter-annular region Ω with $0 < a \le r \le b$ and $0 \le \theta \le \pi/2$. The horizontal face of the plate is insulated while the vertical face is kept at 2 degrees and the outer hoop is maintained at 2 degrees. The inner hoop is maintained at a temperature of $\cos(2\theta)$. Determine the steady state temperature by solving the following BVP in Ω:

$$\begin{cases} v_{rr} + \dfrac{1}{r}v_r + \dfrac{1}{r^2}v_{\theta\theta} = 0 \qquad \text{in } \Omega \\ v_\theta(r,0) = 0 \quad \text{for } a < r < b, \qquad v(r,\pi/2) = 2 \quad \text{for } a < r < b \\ v(a,\theta) = \cos 2\theta \quad \text{for } 0 < \theta < \pi/2, \qquad v(b,\theta) = 2 \quad \text{for } 0 < \theta < \pi/2 \end{cases}$$

Problem 3: Consider the BVP:

$$\begin{aligned} \phi'' + 6\phi' + \lambda\phi &= 0\,, \qquad 0 < x < L \\ \phi(0) &= 0 \\ \phi(L) &= 0 \end{aligned}$$

(a) Put this BVP into Sturm-Liouville form.
(b)Compute all eigenvalues and eigenfunctions.
(c) Show explicitly that the eigenfunctions are mutually orthogonal. (Don't forget to include the weight function inside the integral.)

Problem 4: Consider the eigenvalue problem

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ y(1) &= 0 = y'(2) \end{aligned}$$

a. Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions.
b. Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the quarter-annular region:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2 \\ u(r,0) &= 0 \quad \text{and} \quad \frac{\partial u(r,\pi/2)}{\partial \theta} = f(r) \\ u(1,\theta) &= 0 \quad \text{and} \quad \frac{\partial u(2,\theta)}{\partial r} = 0 \end{aligned}$$

Problem 5:
Solve the following heat conduction problem:

$$\begin{aligned} u_t &= x^2u_{xx} + 4xu_x \quad \text{for } x \in (1,2),\ t > 0 \\ u(1,t) &= 1 \qquad u(2,t) = 1 \\ u(x,0) &= 1 - 5x^{-3/2} \end{aligned}$$

Problem 1:

$u_{tt} - u_{xx} = 0 \quad , \quad 0 \leq x \leq 1 , \; t > 0$

B.C.'s : $\quad u(0,t) = 0 , \quad u(1,t) = 1$

I.C. : $\quad u(x,0) = \sin(\pi x) + x , \quad u_t(x,0) = \sin(3\pi x)$

(a) $\quad W(x) = Ax + B \; , \quad W(0) = 0 = B \; , \quad W(1) = 1 = A$

$\longrightarrow \; W(x) = x$

Decompose the problem:

$u(x,t) = W(x) + v(x,t)$

PDE : $\quad \underbrace{\cancel{W_{tt}}}_{0} + V_{tt} = \underbrace{\cancel{W_{xx}}}_{0} + V_{xx} \quad \rightarrow \quad V_{tt} = V_{xx}$

B.C.'s: $\quad v(0,t) = u(0,t) - W(0,t) = 0$

$\quad v(1,t) = u(1,t) - W(1,t) = 1 - 1 = 0$

I.C.'s: $\quad v(x,0) = u(x,0) - W(x) = \sin(\pi x) + x - x = \sin(\pi x)$

$\quad V_t(x,0) = u_t(x,0) - W_t(x) = \sin(3\pi x) - 0 = \sin(3\pi x)$

So, the B.V.P. for v is :

$$\begin{cases} V_{tt} = V_{xx} & \\ V(0,t) = 0 = V(1,t) & \text{B.C.'s} \\ V(x,0) = \sin(\pi x), \quad V_t(x,0) = \sin(3\pi x) & \text{I.C.} \end{cases}$$

Separation of variables:

$u(x,t) = X(x) \cdot T(t)$

$\rightarrow \quad \ddot{T} X = X'' T \quad \rightarrow \quad \frac{X''}{X} = \frac{\ddot{T}}{T} = \lambda = -\mu^2$

the only non-trivial case

$X] \quad \left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\} \rightarrow \quad X_n = \sin(n\pi x)$

$\mu_n = n\pi \quad , \quad n = 1, 2, 3, \ldots$

$T] \quad \ddot{T} + \mu_n^2 T = 0 \quad \rightarrow \quad T(t) = A_n \cos(\mu_n t) + B_n \sin(\mu_n t)$

So, the general solution is:

$$v(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos(n\pi t) + B_n \sin(n\pi t) \right\} \cdot \sin(n\pi x)$$

Apply the I.C.'s to find A_n's and B_n's:

$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \sin(\pi x) \rightarrow \begin{array}{l} A_1 = 1 \\ A_n = 0 \quad , \text{ if } n \neq 1 \end{array}$$

or: $A_n = \delta_{n1}$

$$v_t(x,0) = \sum_{n=1}^{\infty} B_n \cdot n\pi \sin(n\pi x) = \sin(3\pi x)$$

$\rightarrow \quad B_3 \cdot (3\pi) = 1$

$B_n = 0 \quad \text{if} \quad n \neq 3$

or $\quad B_n = \frac{1}{3\pi} \delta_{n3}$

$$v(x,t) = \sum_{n=1}^{\infty} \left\{ \delta_{n1} \cos(n\pi t) + \frac{1}{3\pi} \delta_{n3} \sin(n\pi t) \right\} \sin(n\pi x)$$

$$= \cos(\pi t) \cdot \sin(\pi x) + \frac{1}{3\pi} \sin(3\pi t) \cdot \sin(3\pi x)$$

$$u(x,t) = x + v(x,t)$$

(b) D'Alembert's solution to the v problem:

$$v(x,t) = \frac{1}{2}\left\{ \overset{o}{f}(x+ct) + \overset{o}{f}(x-ct) \right\} + \frac{1}{2} \int_{x-t}^{x+t} \overset{o}{g}(s) \cdot ds$$

↑ odd extension

$$= \frac{1}{2}\left\{ \sin(\pi(x+t)) + \sin(\pi(x-t)) \right\} + \frac{1}{2} \int_{x-t}^{x+t} \sin(3\pi s)\, ds$$

$$= \frac{1}{2}\left\{ \sin(\pi x)\cos(\pi t) - \cancel{\cos(\pi x)\sin(\pi t)} + \sin(\pi x)\cos(\pi t) + \cancel{\cos(\pi x)\sin(\pi t)} \right\}$$

$$+ \frac{1}{2} \cdot \frac{-1}{3\pi} \left\{ \cos(3\pi(x+t)) - \cos(3\pi(x-t)) \right\}$$

$$= \sin(\pi x)\cos(\pi t)$$

$$- \frac{1}{6\pi} \left\{ \cos(3\pi x) \cdot \cos(3\pi t) - \sin(3\pi x) \sin(3\pi t) - \cos(3\pi x)\cos(3\pi t) - \sin(3\pi x) \sin(3\pi t) \right\}$$

$$= \sin(\pi x) \cos(\pi t) + \frac{1}{3\pi} \sin(3\pi x) \sin(3\pi t)$$

the same as separation of variables

Solving the wave equation with sources and time dependent B.C.'s:

Problem 2: $u_{tt} = c^2 u_{xx} + \underbrace{e^{-t}\sin 5x}_{S(x,t)}$ $\qquad 0 < x < \pi/2$

$t > 0$

time dependent soure

B.C.'s : $u(0,t) = 0, \quad u_x(\pi/2, t) = t$

I.C.'s : $u(x,0) = 0 \quad , \quad u_t(x,0) = \sin 3x + x$

To remove the inhomogeneous time-dependent B.C.'s assume a function $w(x,t)$:

$w(x,t) = A(t)\,x + B(t) \quad , \quad w_x = A(t)$

$w(0,t) = 0 \quad \rightarrow \quad B(t) = 0$

$w_x(\pi/2, t) = A(t) = t$

$\rightarrow \quad w(x,t) = x\,t$

Decompose the problem:

Let $u(x,t) = w(x,t) + v(x,t)$

Find the v- problem.

PDE : $u_{tt} = \underbrace{w_{tt}}_{0} + V_{tt} = c^2(\underbrace{w_{xx}}_{0} + V_{xx}) + e^{-t}\sin(5x)$

$$\rightarrow \quad V_{tt} = c^2 V_{xx} + e^{-t}\sin 5x$$

B.C. :

$$0 = u(0,t) = \underbrace{w(0,t)}_{0} + V(0,t) \quad \rightarrow \quad V(0,t) = 0$$

$$t = u_x(\tfrac{\pi}{2},t) = \underbrace{w_x(\pi/2,t)}_{t} + V_x(\pi/2,t) \quad \rightarrow \quad V_x(\pi/2,t) = 0$$

I.C. :

$$0 = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0)$$

$$\rightarrow \quad V(x,0) = 0$$

$$\sin 3x + x = u_t(x,0) = W_t(x,0) + V_t(x,0)$$

$$= x + V_t(x,0)$$

$$\rightarrow \quad V_t(x,0) = \sin 3x$$

$$\begin{cases} v_{tt} = v_{xx} + e^{-t}\sin 5x \\ v(0,t) = 0 = v_x(\pi/2, t) \\ v(x,0) = 0 \quad , \quad v_t(x,0) = \sin 3x \end{cases}$$

The eigenfunctions and eigenvalues for the homogeneous version of PDE with mixed B.C.'s are:

$$\mu_n = \frac{(2n+1)\pi}{2L} = \frac{(2n+1)\pi}{2\cdot \pi/2} = 2n+1 \qquad n = 1, 2, \ldots$$

$$\lambda_n = -\mu_n^2$$

$$X_n = \sin(\mu_n x)$$

Use the method of eigenfunction expansions for the time dependent source term:

$$S(x,t) = e^{-t}\sin 5x = \sum_{n=1}^{\infty} S_n(t)\sin((2n+1)x)$$

$$\longrightarrow \quad S_n(t) = e^{-t}\delta_{n2} = \begin{cases} e^{-t} & n=2 \\ 0 & n \neq 2 \end{cases}$$

Now expand the solution as eigenfunctions:

$$v(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin(\mu_n x)$$

Substitute into the PDE:

$$V_{tt} - c^2 V_{xx} - e^{-t} \sin 5x = 0$$

$$\sum_{n=1}^{\infty} \left\{ \frac{d^2 V_n}{dt^2} + c^2 \mu_n^2 V_n(t) - \delta_{n2} e^{-t} \right\} \sin(\mu_n x) = 0$$

Linear independency of $\sin(\mu_n x)$ functions gives:

$$\frac{d^2 V_n}{dt^2} + c^2 \mu_n^2 V_n(t) = \delta_{n2} e^{-t} \qquad (*)$$

A 2nd order constant-coefficient ODE:

Solution to the homogeneous ODE is:

$$V_n^c = A \cos(c\mu_n t) + B \sin(c\mu_n t)$$

Finding a particular solution V_n^p:

guess: $V_n^p = c_n e^{-t} \rightarrow \dot{V}_n^p = -c_n e^{-t}$

$$\ddot{V}_n^p = c_n e^{-t}$$

Find C_n by substituting into ODE(*):

$$C_n \cancel{e^{-t}} + c^2\mu_n^2 \cdot C_n \cancel{e^{-t}} = \delta_{n2} \cancel{e^{-t}}$$

$$\rightarrow \quad C_n(1+c^2\mu_n^2) = \delta_{n2}$$

$$C_n = \frac{\delta_{n3}}{1+c^2\mu_n^2}$$

So, the general solution to the ODE (*) is:

$$V_n(t) = V_n^c + V_n^p$$

$$= A\cos(c\mu_n t) + B\sin(c\mu_n t) + \frac{\delta_{n2}}{1+c^2\mu_n^2} e^{-3}$$

Substitute these coefficients into the $V(x,t)$ expansion:

$$V(x,t) = \sum_{n=1}^{\infty} V_n(t) \sin(\mu_n x)$$

$$= \sum_{n=1}^{\infty} \left\{ A_n \cos(c\mu_n t) + B_n \sin(c\mu_n t) + \frac{\delta_{n2} e^{-t}}{1+c^2\mu_n^2} \right\}.$$

$\sin(\mu_n x)$

You can find A_n and B_n by applying the two initial conditions:

First, take time derivative of $v(x,t)$:

$$v_t(x,t) = \sum_{n=1}^{\infty} \left\{ -A_n c\mu_n \sin(c\mu_n t) + B_n c\mu_n \cos(c\mu_n t) - \frac{\delta_{n2}}{1+c^2\mu_n^2} e^{-t} \right\} \cdot \sin(\mu_n x)$$

$$0 = v(x,0) = \sum_{n=1}^{\infty} \left\{ A_n + \frac{\delta_{n2}}{1+c^2\mu_n^2} \right\} \sin(\mu_n x)$$

$$\rightarrow \quad A_n = \frac{-\delta_{n2}}{1+c^2\mu_n^2}$$

$$\sin 3x = v_t(x,0) = \sum_{n=1}^{\infty} \left\{ B_n c\mu_n - \frac{\delta_{n2}}{1+c^2\mu_n^2} \right\} \sin(\mu_n x)$$

Remember: $\mu_n = 2n+1$

$$B_n c\mu_n - \frac{\delta_{n2}}{1+c^2\mu_n^2} = \delta_{n1}$$

$$\rightarrow \quad B_n = \frac{\delta_{n1}}{c\mu_n} + \frac{\delta_{n2}}{c\mu_n(1+c^2\mu_n^2)}$$

Substitute for A_n and B_n :

$$V(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{-\delta_{n2}}{1+c^2\mu_n^2} \cdot \cos(c\mu_n t) + \frac{1}{c\mu_n} \cdot \left(\delta_{n1} + \frac{\delta_{n2}}{1+c^2\mu_n^2}\right) \sin(c\mu_n t) + \frac{\delta_{n2}}{1+c^2\mu_n^2} e^{-t} \right\} \sin(\mu_n x)$$

This sum has only 2 non-zero terms when

$n = 2, 1$:

$n = 2 \longrightarrow \delta_{n2} = 1 , \quad \mu_n = 5$

$n = 1 \longrightarrow \delta_{n1} = 1 , \quad \mu_n = 3$

$$V(x,t) = \frac{1}{1+25c^2} \left[e^{-t} - \cos(5ct) + \frac{1}{5c} \sin(5ct) \right] \sin(5x) + \frac{\sin(3ct)}{3c} \cdot \sin(3x)$$

$$u(x,t) = w(x,t) + V(x+t) \rightarrow \text{The solution}$$

$$w(x,t) = xt$$

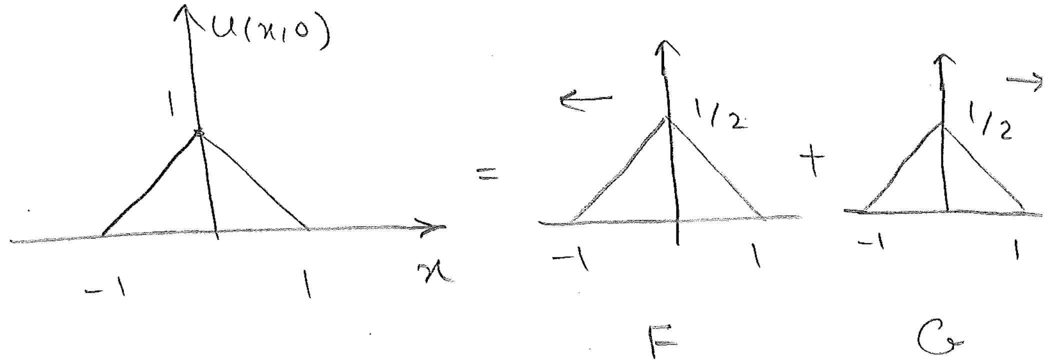
Problem 3:

wave equation:

$$u(x,0) = f(x) = \begin{cases} x+1 & \text{if } -1 \leq x < 0 \\ 1-x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$$u_t(x,0) = 0$$

$$c = \left(\frac{T}{\rho}\right)^{1/2} = 1$$

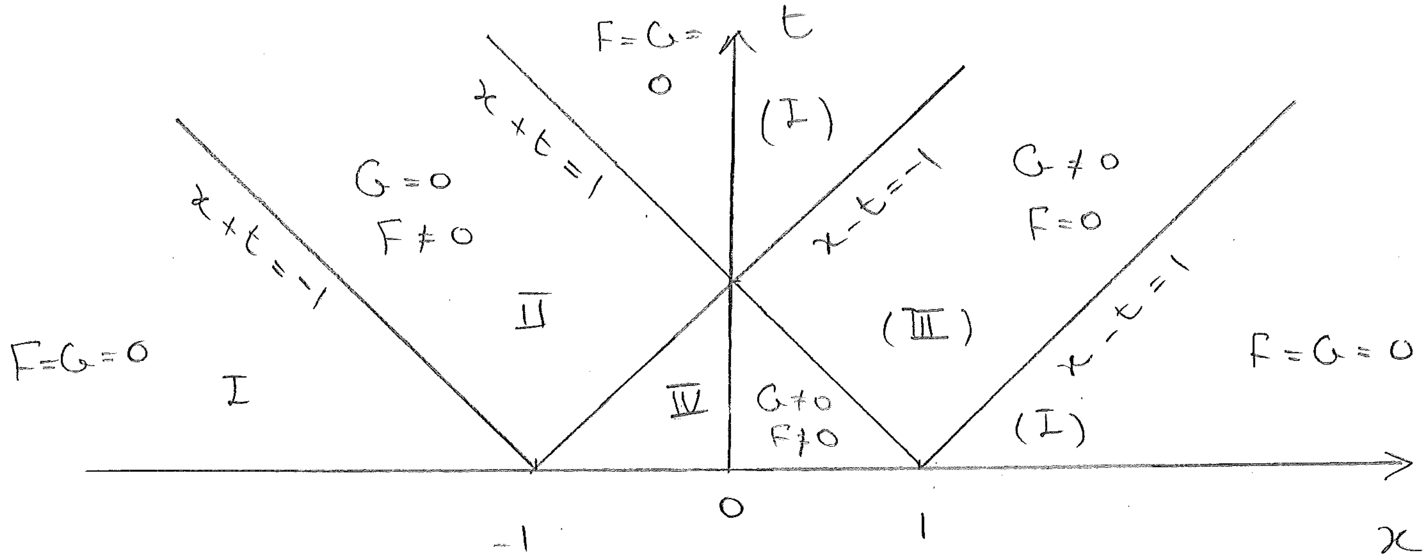


D'Alembert's solution:

$$u(x,t) = \frac{1}{2}\left\{ f(x-t) + f(x+t) \right\} = G(x-t) + F(x+t)$$

$$G(x-t) = \begin{cases} \frac{1}{2}(x-t+1) & \text{if } -1 \leq x-t < 0 \\ \frac{1}{2}(1-x+t) & \text{if } 0 \leq x-t \leq 1 \\ 0 & \text{if } |x-t| > 1 \end{cases}$$

$$F(x+t) = \begin{cases} \frac{1}{2}(x+t+1) & \text{if } -1 \leq x+t < 0 \\ \frac{1}{2}(1-x-t) & \text{if } 0 \leq x+t \leq 1 \\ 0 & \text{if } |x+t| > 1 \end{cases}$$



(I): if $-\infty < x < -t-1$

or $t+1 < x < \infty$

or $1-t < x < -1+t$

then $u(x,t) = 0$

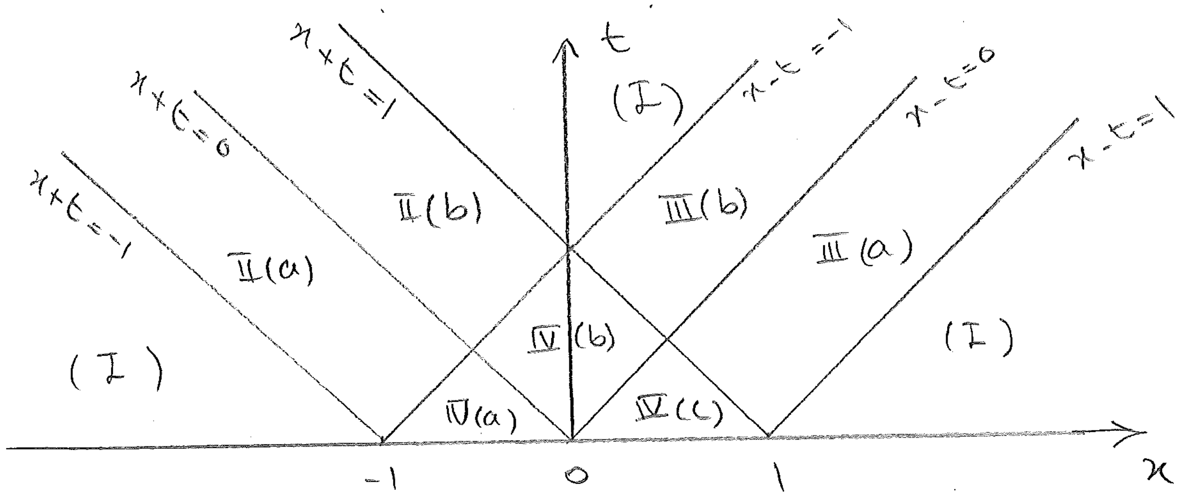
(II): if $-1-t < x < 1-t$ and $-1-t < x < -1+t$, then $u(x,t) = F(x+t)$

(III): if $1-t < x < 1+t$ and $-1+t < x < 1+t$, them

$u(x,t) = G(x-t)$

IV: if $t-1 < x < 1-t$, then $u(x,t) = F(x+t) + G(x-t)$

Or a more detailed identification of regions:



(I): $u(x,t)=0$

II(a): $u(x,t) = \frac{1}{2}(x+t+1)$

II(b): $u(x,t) = \frac{1}{2}(1-x-t)$

III(a): $u(x,t) = \frac{1}{2}(1-x+t)$

III(b): $u(x,t) = \frac{1}{2}(x-t+1)$

IV(a): $u(x,t) = \frac{1}{2}(x-t+1) + \frac{1}{2}(x+t+1)$

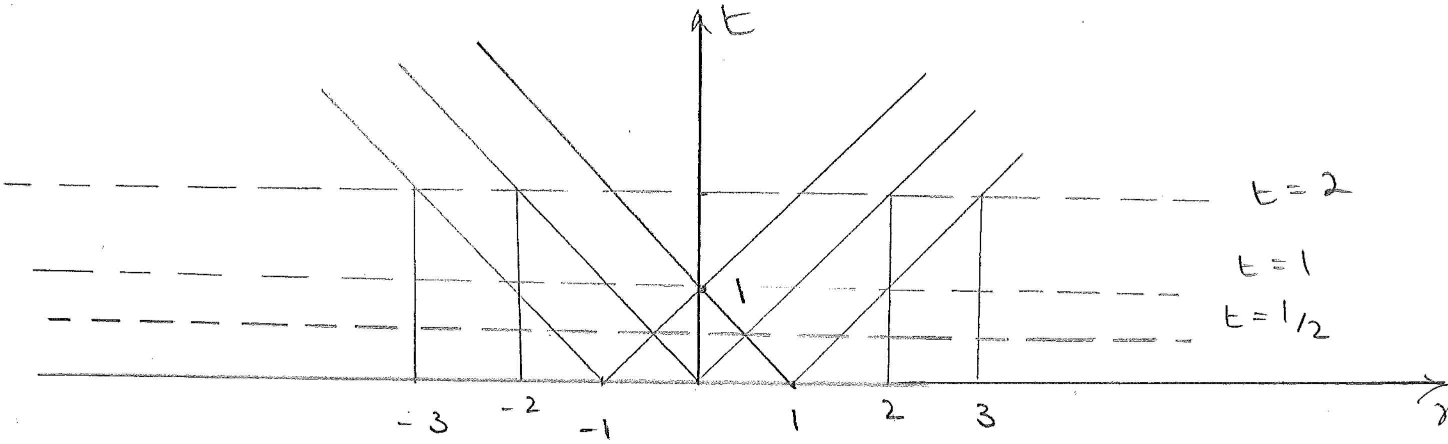
$$= x+1$$

IV(b): $u(x,t) = \frac{1}{2}(x-t+1) + \frac{1}{2}(1-x-t)$

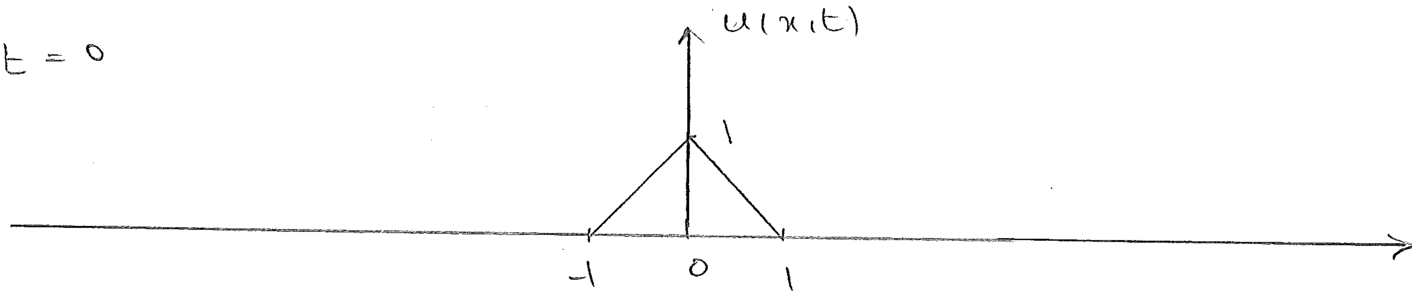
$$= 1-t$$

IV(c): $u(x,t) = \frac{1}{2}(1-x+t) + \frac{1}{2}(1-x-t)$

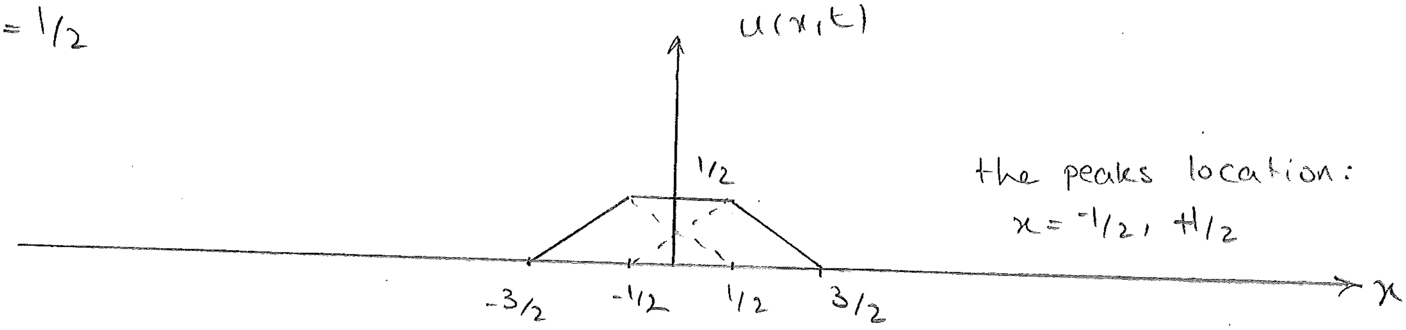
$$= 1-x$$



t = 0

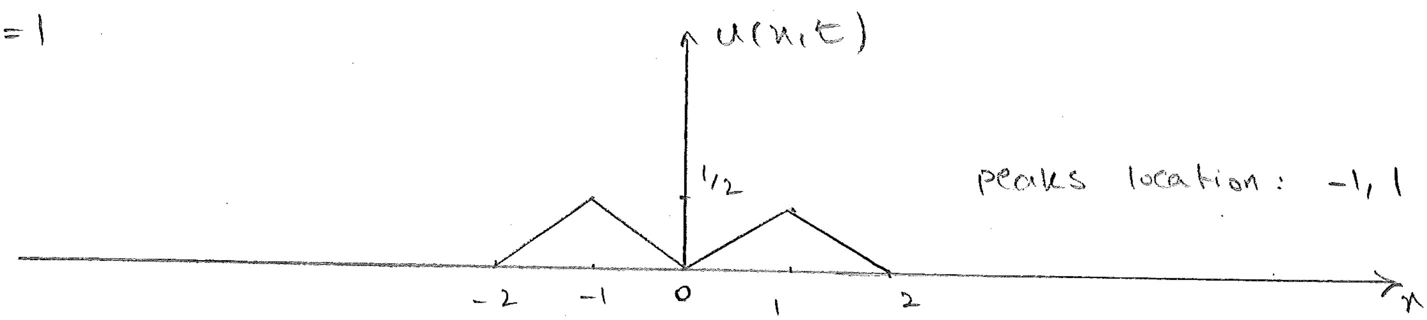


t = 1/2



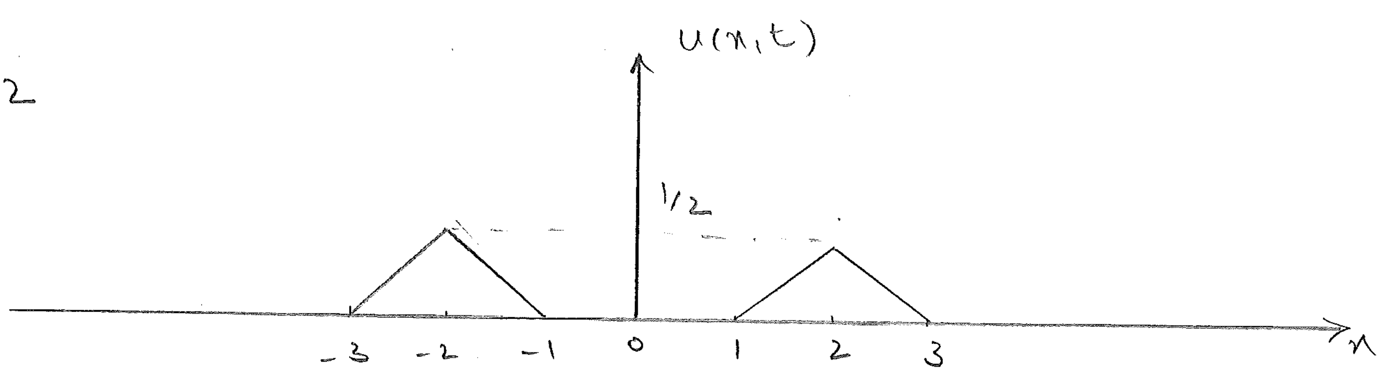
the peaks location:
$x = -1/2, +1/2$

t = 1



peaks location: $-1, 1$

t = 2



Math 257/316 Ass 8 2017 Solutions

Problem 4:

(a) $0 = c^2 u^s_{xx} - g \Rightarrow u^s_{xx} = \frac{g}{c^2}$; $u^s_x = \frac{g}{c^2}x + A$, $u^s = \frac{g}{2c^2}x^2 + Ax + B$

$0 = u^s(0) = B$; $u^s(1) = \frac{g}{2c^2} + A = 0$ $A = -g/2c^2$

$\therefore$ $\boxed{u^s(x) = \frac{g}{2c^2}(x^2 - x)}$

(b) LET $u(x,t) = u^s(x) + v(x,t)$

$\therefore$ $u_{tt} = \cancel{u^s_{tt}} + v_{tt} = c^2(u^s_{xx} + v_{xx}) - g = c^2 v_{xx} + [c^2 \cancel{u^s_{xx}} - g] \Rightarrow$ $v_{tt} = c^2 v_{xx}$

BC $\begin{cases} 0 = u^s(0) + v(0,t) = 0 + v(0,t) \\ 0 = u^s(1) + v(1,t) = 0 + v(1,t) \end{cases}$ $\Rightarrow v(0,t) = 0$, $\Rightarrow v(1,t) = 0$

IC: $\sin \pi x = u(x,0) = u^s(x) + v(x,0)$ $\Rightarrow v(x,0) = \sin(\pi x) - \frac{g}{2c^2}(x^2 - x)$

$0 = u_t(x,0) = \cancel{u^s_t(x)} + v_t(x,0)$ $\Rightarrow v_t(x,0) = 0$

LET $v(x,t) = X(x)T(t) \Rightarrow \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda^2$

X] $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\}$ $\lambda_n = \frac{n\pi}{1}$ $n = 1, 2..$ $X_n = \sin(n\pi x)$

T] $\ddot{T} + \lambda^2 c^2 T = 0$ $T = e^{rt} \Rightarrow T = A\cos\lambda ct + B\sin\lambda ct$

$\therefore$ $v(x,t) = \sum_{n=1}^{\infty} [A_n \cos\lambda_n ct + B_n \sin\lambda_n ct] \sin\lambda_n x$

$v_t(x,t) = \sum_{n=1}^{\infty} [-A_n \lambda_n c \sin\lambda_n ct + B_n \lambda_n c \cos\lambda_n ct] \sin\lambda_n x$.

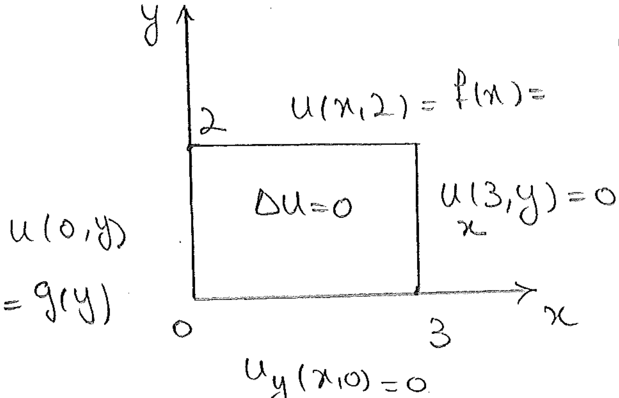
$\sin(\pi x) - \frac{g}{2c^2}(x^2 - x) = v(x,0) = \sum_{n=1}^{\infty} A_n \sin\lambda_n x \Rightarrow A_n = \frac{2}{1}\int_0^1 \{\sin(\pi x) - \frac{g}{2c^2}(x^2 - x)\} \sin n\pi x \, dx$

$\therefore$ $A_n = \delta_{n1} - \frac{g}{c^2}\int_0^1 (x^2 - x)\sin(n\pi x)\,dx = \delta_{n1} - \frac{2g}{c^2}\left[\frac{\cos n\pi - 1}{n^3\pi^3}\right]$

$0 = v_t(x,0) = \sum_{n=1}^{\infty} (B_n \lambda_n c \cdot 1) \sin\lambda_n x \Rightarrow B_n = 0$

$\therefore$ $u(x,t) = \frac{g}{2c^2}(x^2 - x) + \frac{2g}{c^2}\sum_{n=1}^{\infty} \frac{[1 - \cos(n\pi)]}{n^3\pi^3} \cos(n\pi ct)\sin(n\pi x) + \cos(\pi ct)\sin(\pi x)$

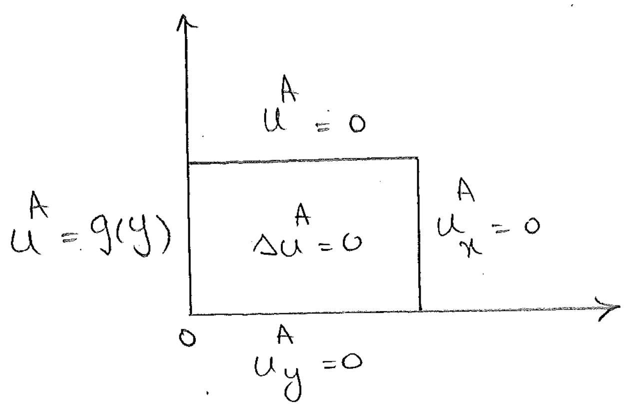
Problem 1:



$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x \le 3 \end{cases}$$

$$g(y) = \begin{cases} y & 0 \le y < 1 \\ 2-y & 1 \le y \le 2 \end{cases}$$

(a) Divide the problem to 2 subproblems A and B:



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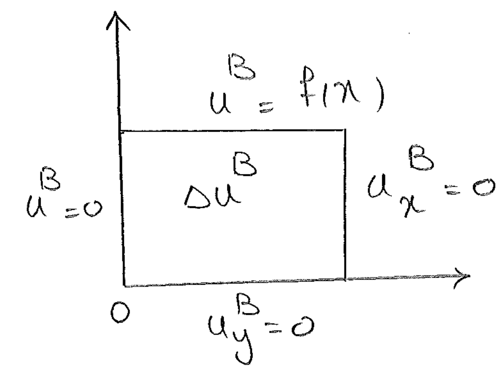


separation of variables:

$$u(x,y) = X(x) \cdot Y(y) \rightarrow X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = \begin{cases} -\mu^2 & \text{eigenvalue problem in } x \\ \mu^2 & \text{eigenvalue problem in } y \end{cases}$$

A – Subproblem A gives an eigen value problem in y:

Y] ODE
$$\left.\begin{aligned} Y'' + \mu^2 Y &= 0 \\ Y'(0) = 0 &= Y(2) \end{aligned}\right\} \rightarrow \begin{aligned} Y_n &= \cos(\mu_n y) \\ \mu_n &= \frac{(2n-1)\pi}{2 \cdot 2} \end{aligned} \quad n = 1, 2, \ldots$$

$$X \Big] \quad X'' - \mu^2 X = 0 \quad \rightarrow \quad X = A \cosh(\mu x) + B \sinh(\mu x)$$

ODE

$$X(3) = 0 \quad \rightarrow \quad A \cancel{\mu} \sinh(3\mu) + B \cancel{\mu} \cosh(3\mu) = 0$$

$$A = -B \frac{\cosh(3\mu)}{\sinh(3\mu)}$$

$$X = \frac{-B}{\sinh(3\mu)} \left[\cosh(\mu x) \cosh(3\mu) - \sinh(\mu x) . \sinh(3\mu) \right]$$

$$= \text{Constant} . \cosh(\mu(x-3))$$

$$u^A(x,y) = \sum_{n=1}^{\infty} A_n \cosh(\mu_n(x-3)) . \cos(\mu_n y)$$

$$\mu_n = \frac{(2n-1)\pi}{4} \qquad n = 1, 2, \dots$$

Finding A_n's :

$$u^A(0,y) = g(y) = \sum_{n=1}^{\infty} \underbrace{A_n \cosh(3\mu_n)}_{a_n^g} \cos(\mu_n y)$$

$$A_n \cosh(3\mu_n) = a_n^g = \frac{2}{2} \int_0^2 g(y) . \cos(\mu_n y) \, dy$$

$$= \int_0^1 y . \cos(\mu_n y) \, dy + \int_1^2 (2-y) \cos(\mu_n y) \, dy$$

$$= y . \frac{1}{\mu_n} \sin(\mu_n y) \Big|_0^1 - \int_0^1 \frac{1}{\mu_n} \sin(\mu_n y) \, dy$$

$$+ \frac{2}{\mu_n} \sin(\mu_n y) \Big|_1^2 - y . \frac{1}{\mu_n} \sin(\mu_n y) \Big|_1^2$$

$$+ \int_1^2 \frac{1}{\mu_n} \sin(\mu_n y) \, dy$$

$$a_n^g = \cancel{\frac{\sin(\mu_n)}{\mu_n}} + \frac{1}{\mu_n^2}\left[\cos(\mu_n) - 1\right] + \cancel{\frac{2}{\mu_n}\sin(2\mu_n)} - \cancel{2\frac{\sin(\mu_n)}{\mu_n}}$$

$$- \cancel{\frac{2}{\mu_n}\sin(2\mu_n)} + \cancel{\frac{\sin(\mu_n)}{\mu_n}} - \frac{1}{\mu_n^2}\left[\cos(2\mu_n) - \cos(\mu_n)\right]$$

Now, note that: $\sin(2\mu_n) = \sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$

$$\cos(2\mu_n) = \cos\left(\frac{(2n-1)\pi}{2}\right) = 0$$

$$a_n^g = -\frac{1}{\mu_n^2} + \frac{2}{\mu_n^2}\cos(\mu_n)$$

$$\rightarrow \quad A_n = \frac{1}{\cosh(3\mu_n)}\left\{\frac{-1}{\mu_n^2} + \frac{2}{\mu_n^2}\cos(\mu_n)\right\}$$

The Solution to problem A is:

$$u^A(x,y) = \sum_{n=1}^{\infty} \frac{a_n^g}{\cosh(3\mu_n)} \cdot \cosh(\mu_n(x-3))\cos(\mu_n y)$$

$$= \sum_{n=1}^{\infty} \frac{(2\cos(\mu_n)-1)}{\mu_n^2 \cdot \cosh(3\mu_n)} \cosh(\mu_n(x-3))\cos(\mu_n y)$$

$$\mu_n^A = \frac{(2n-1)\pi}{4}, \quad n = 1, 2, \cdots$$

B - Subproblem B gives an eigenvalue problem in x direction :

$$X] \quad \left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X(0) = 0 = X(3) \end{array}\right\} \rightarrow \quad \begin{array}{l} X_n = \sin(\mu_n x) \\ \mu_n = \dfrac{(2n-1)\pi}{3.2} \end{array} \quad n = 1, 2, \ldots$$

$$X] \quad Y'' - \mu^2 Y = 0 \quad \rightarrow \quad Y = C\cosh(\mu y) + D\sinh(\mu y)$$

$$Y'(0) = 0 \quad \rightarrow \quad C.\mu \sinh(\mu.0) + D\mu\cosh(\mu.0) = 0$$

$$\text{or :} \quad D = 0$$

$$\text{So :} \quad Y_n = \cosh(\mu_n y)$$

$$u^B(x,y) = \sum_{n=1}^{\infty} B_n \cosh(\mu_n y).\sin(\mu_n x)$$

Find the B_n's by using the non-zero B.C. :

$$u^B(x,2) = f(x) = \sum_{n=1}^{\infty} \underbrace{B_n \cosh(2\mu_n)}_{b_n^f} \sin(\mu_n x)$$

$$b_n^f = \frac{2}{3}\int_0^3 f(x).\sin(\mu_n x)\,dx$$

$$= \frac{2}{3}\left[\int_0^1 x.\sin(\mu_n x)\,dx + \int_1^3 (2-x)\sin(\mu_n x)\,dx\right]$$

$$b_n^f = \frac{2}{3}\left[\, x \cdot \frac{-1}{\mu_n}\cos(\mu_n x)\Big|_0^1 + \int_0^1 \frac{1}{\mu_n}\cos(\mu_n x)\,dx \right.$$

$$\left. + \, 2 \cdot \frac{-1}{\mu_n}\cos(\mu_n x)\Big|_1^3 + \frac{x}{\mu_n}\cos(\mu_n x)\Big|_1^3 - \int_1^3 \frac{1}{\mu_n}\cos(\mu_n x)\,dx \right]$$

$$= \frac{2}{3}\left[\, \cancel{-\frac{1}{\mu_n}\cos(\mu_n)} + \frac{1}{\mu_n^2}\sin(\mu_n) - \frac{2}{\mu_n}\cos(3\mu_n) + \cancel{\frac{2}{\mu_n}\cos(\mu_n)} \right.$$

$$\left. + \frac{3}{\mu_n}\cos(3\mu_n) - \cancel{\frac{1}{\mu_n}\cos(\mu_n)} - \frac{1}{\mu_n^2}\sin(3\mu_n) + \frac{1}{\mu_n^2}\sin(\mu_n) \right]$$

$$= \frac{2}{3}\left[\, \frac{2}{\mu_n^2}\sin(\mu_n) + \frac{1}{\mu_n}\cos(3\mu_n) - \frac{1}{\mu_n}\sin(3\mu_n) \right]$$

Note: $\cos(3\mu_n) = \cos\left(3 \cdot \frac{(2n-1)\pi}{3 \cdot 2}\right) = 0$

$\sin(3\mu_n) = \sin\left(\frac{(2n-1)}{2}\pi\right) = (-1)^{n+1}$

$$\rightarrow \quad b_n^f = \frac{4\sin(\mu_n)}{3\mu_n^2} - \frac{2(-1)^{n+1}}{3\mu_n^2}$$

$B_n = \dfrac{b_n^f}{\cosh(2\mu_n)}$, so the solution to the B Problem is:

$$u^B(x,y) = \sum_{n=1}^{\infty} \frac{b_n^f}{\cosh(2\mu_n)} \cosh(\mu_n y)\sin(\mu_n x)$$

or :

cosh(\mu_n*y) sin(\mu_n*x)

$$u^B(x,y) = \sum_{n=1}^{\infty} \frac{4\sin(\mu_n) + 2(-1)^n}{3\mu_n^2 \cosh(2\mu_n)} \cdot \cancel{\cosh(\mu_n(x-3))} \cdot \cancel{\cos(\mu_n y)}$$

$$\mu_n^B = \frac{(2n-1)\pi}{6}$$

The solution to the full problem is:

$$u(x,y) = u^A(x,y) + u^B(x,y)$$

(b) verify the B.C.'s:

$$u(x,y) = \sum_{n=1}^{\infty} \frac{a_n^g}{\cosh(3\mu_n)} \cosh(\mu_n^A(x-3)) \cos(\mu_n^A y)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n^f}{\cosh(2\mu_n^B)} \cosh(\mu_n^B y) \cdot \sin(\mu_n^B x)$$

B.C.1:

$$u_x(3,y) = \sum_{n=1}^{\infty} \frac{a_n^g}{\cosh(3\mu_n^A)} \cdot \mu_n \underbrace{\sinh(\mu_n^A \cdot 0)}_{0} \cos(\mu_n^A y)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n^f}{\cosh(2\mu_n^B)} \cosh(\mu_n^B y) \cdot \mu_n \cdot \underbrace{\cos(3\mu_n^B)}_{= \cos\left(\frac{(2n-1)\pi}{2}\right) = 0}$$

$$= 0 + 0 = 0$$

B.C. 2:

$$u_y(x,0) = \sum_{n=1}^{\infty} \frac{a_n^g}{\cosh(3\mu_n^A)} \cosh(\mu_n^A(x-3))\cdot(-\mu_n)\cdot\underbrace{\sin(\mu_n^A\cdot 0)}_{0}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n^f}{\cosh(2\mu_n^B)} \mu_n \underbrace{\sinh(\mu_n^B y)}_{0}\cdot \sin(\mu_n^B x)$$

$$= 0 + 0 = 0$$

B.C. 3:

$$u(x,2) = \sum_{n=1}^{\infty} \frac{a_n^g}{\cosh(3\mu_n^A)} \cdot \cosh(\mu_n^A(x-3))\ \underbrace{\cos(\mu_n^A\cdot 2)}_{=\cos\left(\frac{(2n-1)\pi}{2}\right)=0}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n^f}{\cancel{\cosh(2\mu_n^B)}} \cdot \cancel{\cosh(\mu_n^B\cdot 2)}\cdot \sin(\mu_n^B x)$$

$$= 0 + \sum_{n=1}^{\infty} b_n^f \sin(\mu_n^B x) = f(x)$$

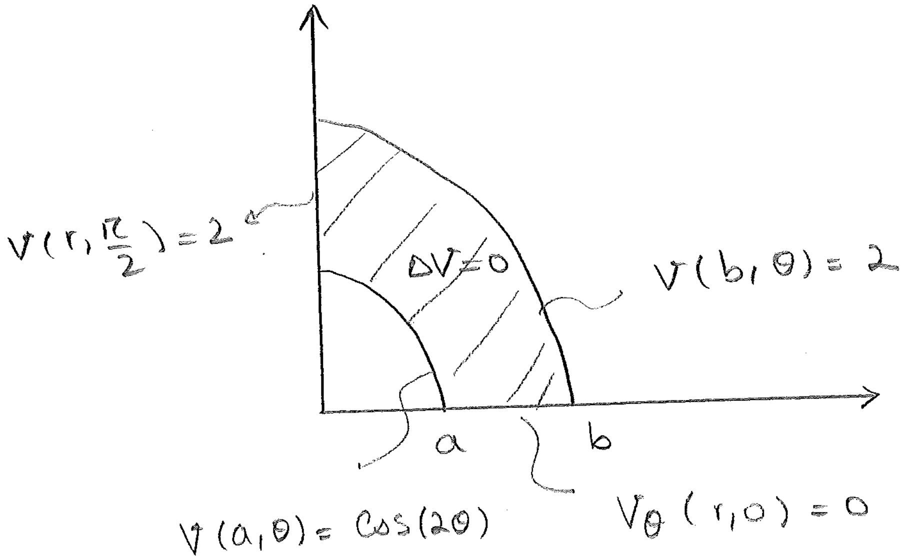
B.C. 4:

$$u(0,y) = \sum_{n=1}^{\infty} \frac{a_n^g}{\cancel{\cosh(3\mu_n^A)}} \cdot \cancel{\cosh(3\mu_n^A)} \cos(\mu_n y)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n^f}{\cosh(2\mu_n^B)} \cdot \cosh(\mu_n^B y)\cdot \underbrace{\sin(\mu_n^B\cdot 0)}_{0}$$

$$= \sum_{n=1}^{\infty} a_n^g \cos(\mu_n y) + 0 = g(y)$$

Problem 2:



gives an eigenvalue problem in θ

$$\begin{cases} v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}=0 \\ v_\theta(r,0)=0 \quad \text{for} \quad a<r<b, \; v(r,\frac{\pi}{2})=2, \; \text{for } a<r<b \\ v(a,\theta)=\cos(2\theta) \quad \text{for} \quad 0<\theta<\frac{\pi}{2}, \; v(b,\theta)=2 \quad \text{for } 0<\theta<\frac{\pi}{2} \end{cases}$$

First, remove the inhomogeneous B.C.:

$W(\theta)=A\theta+B, \quad W'(0)=0, \; W(\frac{\pi}{2})=2 \rightarrow A=0, \; B=2$

$W(\theta)=2$

Now, divide the solution to two parts:

$v(r,\theta)=W(\theta)+u(r,\theta)$

Find the B.V.P. for $u(r,\theta)$:

$\underbrace{\Delta W}_{0}+\Delta u=0 \rightarrow \Delta u=0$ PDE

B.C. 1: $v_\theta(r,0)=0=\underbrace{W_\theta(0)}_{0}+u_\theta(r,0) \rightarrow u_\theta(r,0)=0$

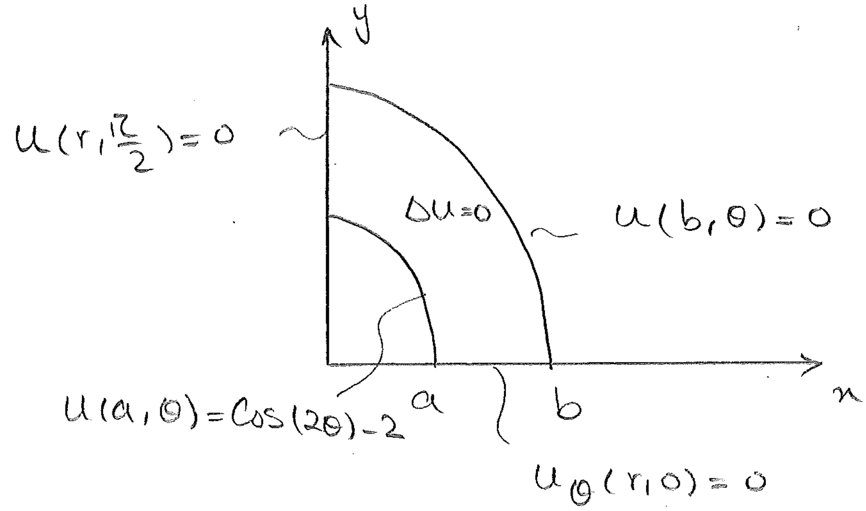
B.C. 2: $v(r,\frac{\pi}{2}) = 2 = \underbrace{w(\frac{\pi}{2})}_{2} + u(r,\frac{\pi}{2}) \rightarrow u(r,\frac{\pi}{2}) = 0$

B.C. 3: $v(a,\theta) = \cos(2\theta) = w(\theta) + u(a,\theta) \rightarrow u(a,\theta) = \cos(2\theta) - 2$

B.C. 4: $v(b,\theta) = 2 = w(\theta) + u(b,\theta) \rightarrow u(b,\theta) = 0$

The new B.V.P. is:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$



Solve by separation of variables:

$$u(r,\theta) = R(r) \cdot \Theta(\theta)$$

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R \cdot \Theta'' = 0$$

Multiply by $\frac{r^2}{R\Theta}$: $\quad \frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda = \underbrace{\mu^2}$

eigenvalue problem in θ

$\Theta]$ $\left.\begin{array}{l} \Theta'' + \mu^2\Theta = 0 \\ \Theta'(0) = 0 = \Theta(\frac{\pi}{2}) \end{array}\right\} \rightarrow \Theta_n = \cos(\mu_n \theta)$

$$\mu_n = \frac{(2n-1)\cancel{\pi}}{\cancel{2} \cdot \frac{\cancel{\pi}}{\cancel{2}}} = 2n-1$$

$n = 1, 2, \ldots$

R] $r^2 R'' + r R' - \mu^2 R = 0$

C-E equation, guess: $R(r) = r^\gamma$, $R'(r) = \gamma r^{\gamma-1}$, $R''(r) = \gamma(\gamma-1) r^{\gamma-2}$

$$\rightarrow \quad r^\gamma \left[\gamma(\gamma-1) + \gamma - \mu^2 \right] = 0 \quad \text{or} \quad \gamma^2 \cancel{- \gamma + \gamma} - \mu^2 = 0$$

$$\gamma = \pm \mu$$

$$R(r) = A r^{-\mu} + B r^{+\mu}$$

So, the solution is:

$$u(r,\theta) = \sum_{n=1}^{\infty} \left\{ A_n r^{-\mu_n} + B_n r^{\mu_n} \right\} \cos(\mu_n \theta)$$

where: $\mu_n = 2n-1 \qquad n = 1, 2, \dots$

$$u(b,\theta) = 0 \quad \rightarrow \quad u(b,\theta) = \sum_{n=1}^{\infty} \left\{ A_n b^{-\mu_n} + B_n b^{\mu_n} \right\} \cos(\mu_n \theta) = 0$$

$$\text{or:} \quad A_n = -B_n b^{2\mu_n}$$

So, the solution simplifies to:

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n \left\{ -b^{2\mu_n} \cdot r^{-\mu_n} + r^{\mu_n} \right\} \cos(\mu_n \theta)$$

The last B.C. is:

$$u(a,\theta) = \cos(2\theta) - 2 = f(\theta)$$

$$\rightarrow \quad B_n \left\{ -b^{2\mu_n} a^{-\mu_n} + a^{\mu_n} \right\} = a_n^{f(\theta)}$$

$$B_n b^{\mu_n} \left\{ -\left(\frac{b}{a}\right)^{\mu_n} + \left(\frac{a}{b}\right)^{\mu_n} \right\} = a_n^{f(\theta)}$$

$$\rightarrow B_n = \frac{b^{-\mu_n}\, a_n^{f(\theta)}}{\left(\frac{a}{b}\right)^{\mu_n} - \left(\frac{b}{a}\right)^{\mu_n}}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} a_n^{f(\theta)} \cdot \frac{\left\{ \left(\frac{r}{b}\right)^{\mu_n} - \left(\frac{b}{r}\right)^{\mu_n} \right\}}{\left(\frac{a}{b}\right)^{\mu_n} - \left(\frac{b}{a}\right)^{\mu_n}} \cos(\mu_n \theta)$$

Now, find $a_n^{f(\theta)}$:

$$a_n^{f(\theta)} = \frac{2}{\pi/2} \int_0^{\pi/2} \left[\cos(2\theta) - 2\right] \cdot \cos((2n-1)\theta)\, d\theta$$

$$= \frac{4}{\pi} \left\{ \int_0^{\pi/2} \frac{1}{2} \left[\cos((2n-3)\theta) + \cos((2n+1)\theta) \right] - 2 \int_0^{\pi/2} \cos((2n-1)\theta)\, d\theta \right\}$$

$$= \frac{4}{\pi} \left\{ \frac{1}{2} \left[\frac{\sin((2n-3)\theta)}{2n-3} + \frac{\sin((2n+1)\theta)}{2n+1} \right]_0^{\pi/2} - \frac{2}{(2n-1)} \sin((2n-1)\theta) \Big|_0^{\pi/2} \right\}$$

$$= \frac{4}{\pi} \left\{ \frac{1}{2} \frac{\sin\left((2n-3)\frac{\pi}{2}\right)}{2n-3} + \frac{1}{2} \frac{\sin\left((2n+1)\frac{\pi}{2}\right)}{2n+1} - 2\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{2n-1} \right\}$$

Note:
$$\begin{cases} \sin\left((2n+1)\frac{\pi}{2}\right) = \sin\left((2n-3)\frac{\pi}{2}\right) = (-1)^n \\ \sin\left((2n-1)\frac{\pi}{2}\right) = (-1)^{n+1} \end{cases}$$

$$\xrightarrow{f(\theta)} a_n = \frac{2}{\pi}\left\{\frac{4n-2}{(2n+1)(2n-3)}\right\}(-1)^n - \frac{8}{\pi}\cdot\frac{(-1)^{n+1}}{(2n-1)}$$

The final solution is ;

$$V(r,\theta) = 2 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{\pi}\left\{\frac{2n-1}{(2n+1)(2n-3)} + \frac{2}{(2n-1)}\right\};$$

$$\frac{\left(\frac{r}{b}\right)^{2n-1} - \left(\frac{b}{r}\right)^{2n-1}}{\left(\frac{a}{b}\right)^{2n-1} - \left(\frac{b}{a}\right)^{2n-1}} \cdot \cos((2n-1)\theta)$$

problem 3: $L\phi = -\phi'' - 6\phi' = \lambda\phi \qquad 0<x<L$

$\phi(0) = 0 = \phi(L)$

a) $F(x) = \frac{e^{\int \frac{6}{1}dx}}{1} = e^{6x}$

$\mathcal{L}\phi = FL\phi = -e^{6x}\phi'' - 6e^{6x}\phi' = \lambda e^{6x}\phi.$

$\therefore \mathcal{L}\phi = -(e^{6x}\phi')' = \lambda e^{6x}\phi$

b) HOMOGENEOUS EQ $\phi'' + 6\phi' + \lambda\phi = 0$

$\phi = e^{rx} \Rightarrow r^2 + 6r + \lambda = 0$

$r = \frac{-6 \pm \sqrt{6^2 - 4\lambda}}{2} = -3 \pm \sqrt{9-\lambda} = -3 \pm \delta \qquad \delta = \sqrt{9-\lambda}$

$\lambda < 9$: $\phi = Ae^{(-3+\delta)x} + Be^{(-3-\delta)x}$

$\phi(0) = A + B = 0 \Rightarrow B = -A$

$\phi(L) = Ae^{-3L}(e^{\delta L} - e^{-\delta L}) = 2Ae^{-3L}\sinh(\delta L) = 0$

SINCE $\delta \neq 0 \neq L$ THE ONLY SOL'N IS THE TRIVIAL SOL'N

$\lambda = 9$: $\phi = Ae^{-3x} + Bxe^{-3x}$

$\phi(0) = A = 0$

$\phi(L) = BLe^{-3L} = 0 \Rightarrow B = 0$ TRIVIAL SOLUTION

$\lambda > 9$: LET $r = -3 \pm \sqrt{\lambda - 9}\, i = -3 \pm i\mu$ WHERE $\mu = \sqrt{\lambda-9}$

THEN $\phi(x) = e^{-3x}[A\cos\mu x + B\sin\mu x]$ $\qquad \lambda = 9 + \mu^2$

$0 = \phi(0) = A \Rightarrow A = 0$

$0 = \phi(L) = e^{-3L}B\sin\mu L = 0 \qquad \mu L = n\pi \quad n = 1, 2, \ldots$

EIGENVALUES ARE: $\lambda = 9 + (n\pi)^2 \quad n = 1, 2, \ldots$

& EIGENFUNCTIONS $\phi_n(x) = e^{-3x}\sin\mu_n x$

(c) $\int_0^1 e^{6x}\phi_m(x)\phi_n(x)\,dx = \int_0^1 e^{6x}\{e^{-3x}\sin(m\pi x)\}\{e^{-3x}\sin(n\pi x)\}\,dx$

$= \int_0^1 \sin(m\pi)x \, \sin(n\pi x)\,dx = 0$

Problem 4:

4. Consider the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0 \qquad (1)$$
$$y(1) = 0 = y'(2)$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the quarter-annular region:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < 2, \quad 0 < \theta < \pi/2$$
$$u(r,0) = 0 \text{ and } \frac{\partial u(r,\pi/2)}{\partial \theta} = f(r)$$
$$u(1,\theta) = 0 \text{ and } \frac{\partial u(2,\theta)}{\partial r} = 0$$

[12 marks]
[total 20 marks]

a) $W(x) = \dfrac{e^{\int \frac{x}{x^2}dx}}{x^2} = \dfrac{e^{\ln x}}{x^2} = \dfrac{1}{x}$ SO MULTIPLY (1) BY $W(x)$ TO OBTAIN

$\mathcal{L}y = -(xy')' = \lambda \frac{1}{x} y$ WHICH IS IN S-L FORM.

LET $y = x^\gamma \Rightarrow \gamma(\gamma-1) + \gamma + \lambda = \gamma^2 + \lambda = \gamma^2 + \mu^2 = 0 \Rightarrow \gamma = \pm i\mu$ WHERE $\lambda = \mu^2$

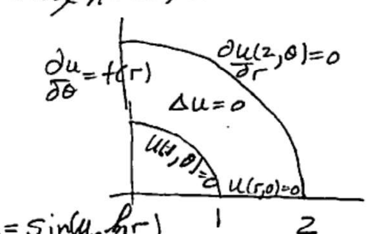
$\therefore\ y(x) = A\cos(\mu \ln x) + B\sin(\mu \ln x) \quad y' = -A\mu(\sin \mu \ln x)\frac{1}{x} + B\mu(\cos \mu \ln x)\frac{1}{x}$

$0 = y(1) = A \quad 0 = y'(2) = B\frac{\mu}{2}\cos(\mu \ln 2) \Rightarrow \mu_n = \dfrac{(2n+1)\pi}{2\ln 2} \quad n = 0, 1, \ldots$ ARE EIGENVALUES

& $y_n(x) = \sin(\mu_n \ln x)$ ARE THE EIGENFCNS.

b) LET $u(r,\theta) = R(r)\Theta(\theta)$ THEN

$\dfrac{r^2R'' + rR'}{R(r)} = -\dfrac{\Theta''}{\Theta(\theta)} = -\lambda = -\mu^2$ CONSTANT



$\Theta]$ $\left.\begin{array}{l}\Theta'' - \mu^2\Theta = 0 \\ \Theta(0) = 0\end{array}\right\}$ $\Theta = A\cosh \mu\theta + B\sinh \mu\theta$
$0 = \Theta(0) = A \Rightarrow \Theta = B\sinh \mu\theta$

$R]$ $\left.\begin{array}{l}r^2R'' + rR' + \mu^2 R = 0 \\ R(1) = 0 = R'(2)\end{array}\right\} \Rightarrow \mu_n = \dfrac{(2n+1)\pi}{2\ln 2} \quad n = 0, 1, \ldots \quad R_n = \sin(\mu_n \ln r)$

$\therefore\ u(r,\theta) = \sum_{n=0}^{\infty} B_n \sinh \mu_n\theta \, \sin(\mu_n \ln r)$

$\dfrac{\partial u}{\partial \theta} = \sum_{n=0}^{\infty} B_n\mu_n \cosh(\mu_n\theta)\sin \mu_n \ln r$

LAST BC: $f(r) = \dfrac{\partial u}{\partial \theta}(r, \tfrac{\pi}{2}) = \sum_{n=0}^{\infty} \overbrace{B_n\mu_n\cosh(\mu_n \tfrac{\pi}{2})}^{d_n}\sin(\mu_n \ln r) = \sum_{n=0}^{\infty} d_n \sin(\mu_n \ln r)$

THEN $\int_1^2 \frac{1}{r} f(r)\sin(\mu_m \ln r)\,dr = \sum_{n=0}^{\infty} d_n \int_1^2 \frac{1}{r}\sin(\mu_m \ln r)\sin(\mu_n \ln r)\,dr$

let $x = \ln r \quad dx = \frac{dr}{r}$ $\qquad = \sum_{n=0}^{\infty} d_n \underbrace{\int_0^{\ln 2} \sin(\mu_m x)\sin(\mu_n x)\,dx}_{\delta_{mn}\ln 2/2}$

$= d_m \ln 2/2.$

$\therefore\ d_m = B_m\mu_m\cosh(\mu_m \tfrac{\pi}{2}) = \dfrac{2}{\ln 2}\int_1^2 \frac{1}{r} f(r)\sin(\mu_m \ln r)\,dr$

$\therefore\ B_m = \dfrac{2}{\ln 2\,\mu_m\cosh(\mu_m \pi/2)}\int_1^2 \frac{1}{r} f(r)\sin(\mu_m \ln r)\,dr$

Problem 5: $u_t = x^2 u_{xx} + 4x u_x \quad 1<x<2 \quad t>0$

$u(1,t)=1 \qquad u(2,t)=1$

$u(x,0) = 1-5x^{-3/2}$

LOOK FOR A STEADY SOLUTION $W(x)$: $x^2 W_{xx} + 4x W_x = 0$

$W = x^r \Rightarrow r(r-1)+4r = r^2+3r = r(r+3)=0 \qquad r=0,-3$

$W = A + Bx^{-3}$

$W(1) = A+B = 1 \qquad A=(1-B)$

$W(2) = \cancel{1}-B+B2^{-3} = \cancel{1} \Rightarrow B(1/8-1)=0 \qquad B=0$

$\therefore W(x)=1$ IS A STEADY SOL'N — THIS CAN BE SEEN BY INSPECTION

LET $u(x,t) = W(x)+V(x,t)$

$u_t = (\cancel{W}+V)_t = x^2u_{xx}+4xu_x = (\cancel{x^2W_{xx}+4xW_x}) + x^2V_{xx}+4xV_{xx}$

$\therefore \quad V_t = x^2V_{xx}+4xV_x$

$\cancel{1} = u(1,t) = W(1)+V(1,t) = \cancel{1}+V(1,t) \Rightarrow V(1,t)=0$

$1 = u(2,t) = W(2)+V(2,t) = 1+V(2,t) \Rightarrow V(2,t)=0$

$1-5x^{-3/2} = u(x,0) = 1+V(x,0) \Rightarrow \qquad V(x,0) = -5x^{-3/2}$.

NOW LET $V(x,t) = X(x)T(t) \Rightarrow X\dot{T} = (x^2X''+4xX')T$

$\therefore \quad \dot{T}/T(t) = (x^2X''+4xX')/X = -\lambda$

$\boxed{T}\quad \dot{T} = -\lambda T \Rightarrow T(t) = Ce^{-\lambda t}$

$\boxed{X}\quad x^2X''+4xX'+\lambda X = 0 \qquad X(1)=0=X(2)$

LET'S WRITE IN S-L FORM $F(x) = e^{\int \frac{4x}{x^2}dx}/x^2 = e^{4\ln x}/x^2 = x^2$

$x^4X''+4x\,x^3X'+\lambda x^2 X = (x^4X')'+\lambda x^2 X = 0.$

HOMOG. CE:EQ $\Rightarrow$ LET $X = x^\gamma \Rightarrow \gamma(\gamma-1)+4\gamma+\lambda = \gamma^2+3\gamma+\lambda=0$

$\therefore \quad \gamma = \frac{-3\pm\sqrt{9-4\lambda}}{2} = -\frac{3}{2}\pm\frac{1}{2}\sqrt{9-4\lambda}$

$\lambda < 9/4$: $\quad X = x^{-3/2}(C_1x^\theta + C_2x^{-\theta}) \qquad \theta = \frac{1}{2}\sqrt{9-4\lambda}$

$X(1) = C_1+C_2 = 0 \qquad C_2=-C_1$

$X(2) = 2^{-3/2}C_1(2^\theta - 2^{-\theta}) = 0$

EITHER $C_1=0$ OR $2^\theta-2^{-\theta}=0 \Rightarrow 2^{2\theta}=1 \Rightarrow 2\theta\ln 2=0 \Rightarrow \theta=0$ WHICH CONTRADICTS $\lambda<9/4$

$\therefore C_1=0$ SO THE ONLY SOL'N IS THE TRIVIAL SOLN

$\lambda = \frac{9}{4}$: $X = c_1 x^{-3/2} + c_2 x^{-3/2} \ln x$

$X(1) = c_1 = 0$

$X(2) = c_2 2^{-3/2} \ln 2 = 0 \Rightarrow c_2 = 0$

ONLY SOL'N IS THE TRIVIAL SOL'N.

$\lambda > \frac{9}{4}$: LET $\gamma = -\frac{3}{2} \pm \frac{i}{2}\sqrt{4\lambda - 9} = -\frac{3}{2} \pm i\mu$ WHERE $\mu = \frac{1}{2}\sqrt{4\lambda-9}$

OR $4\mu^2 = 4\lambda - 9 \Rightarrow \boxed{\lambda = \frac{9}{4} + \mu^2}$

$\therefore \; X = x^{-3/2}[A\cos(\mu \ln x) + B\sin(\mu \ln x)]$

$X(1) = A\cos(\mu \ln 1) = A = 0$

$X(2) = B\, 2^{-3/2} \sin(\mu \ln 2) = 0 \Rightarrow \mu_n = \frac{n\pi}{\ln 2} \quad n = 1, 2, \ldots$

$X_n = x^{-3/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$

$\therefore \; V(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} x^{-3/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right) \qquad \lambda_n = \frac{9}{4} + \left(\frac{n\pi}{\ln 2}\right)^2$

$V(x,0) = -5x^{-3/2} = \sum_{n=1}^{\infty} c_n x^{-3/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$

$\therefore \; -5\int_1^2 x^2 (x^{-3/2})\, x^{-3/2} \sin(\mu_m \ln x)\, dx = \int_1^2 x^2 \left\{\sum_{n=1}^{\infty} c_n x^{-3/2} \sin \mu_n \ln x\right\} x^{-3/2} \sin(\mu_m \ln x)\, dx$

$-5\int_1^2 x^{-1} \sin(\mu_m \ln x)\, dx = \sum_{n=1}^{\infty} c_n \int_1^2 x^{-1} \sin(\mu_n \ln x) \sin(\mu_m \ln x)\, dx$

LET $\chi = \ln x \quad d\chi = dx/x \quad x = 1 \Rightarrow \chi = 0 \quad x = 2 \Rightarrow \chi = \ln 2$.

$-5\int_0^{\ln 2} \sin(\mu_m \chi)\, d\chi = \sum_{n=1}^{\infty} c_n \int_0^{\ln 2} \sin(\mu_n \chi) \sin(\mu_m \chi)\, d\chi$

$5\, \frac{\cos\left(\frac{n\pi}{\ln 2}\chi\right)\Big|_0^{\ln 2}}{\mu_m} = \sum_{n=1}^{\infty} c_n \delta_{mn} \frac{\ln 2}{2}$

$\therefore \; \frac{5[\cos(m\pi) - 1]}{\mu_m} = c_m \frac{\ln 2}{2} \Rightarrow c_n = \frac{10}{\ln 2} \frac{[(-1)^n - 1]}{(n\pi)/\ln 2} = \frac{10[(-1)^n - 1]}{(n\pi)}$

$\therefore \; u(x,t) = 1 + \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n} e^{-\left[\frac{9}{4} + \left(\frac{n\pi}{\ln 2}\right)^2\right]t} x^{-3/2} \sin\left(\frac{n\pi \ln x}{\ln 2}\right)$