Mathematics 317 — Final Exam — 180 minutes

April 22, 2025

- The test consists of 13 pages and 8 questions worth a total of 61 marks.
- This is a closed-book examination. You are allowed to bring in and use one formula sheet (this sheet should be reasonable: for example you shouldn't need special glasses or intsruments to read it). **None of the following are allowed**: documents, formula sheets other than the one you came with, or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Signature								
Name	Solutions							

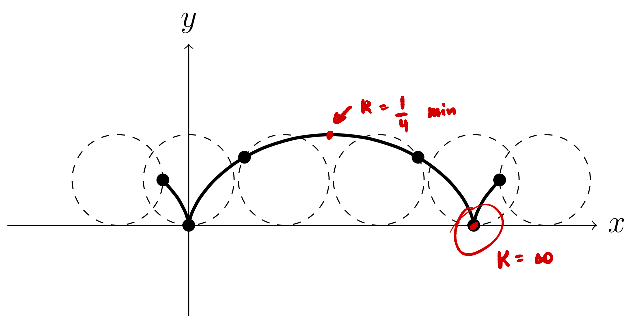
Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. The cycloid is the trajectory travelled by an object attached to the rim of a bicycle tire where the bicycle is travelling at constant speed. The cycloid is given by the vector valued function

$$\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle .$$



(a) $\boxed{\text{2 marks}}$ Find and simplify $|\vec{r}\,'(t)|$.

$\vec{r}\,'(t) = \langle 1 - \cos t, \sin t \rangle$

$|\vec{r}\,'(t)| = \sqrt{(1-\cos t)^2 + \sin^2 t}$

$= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}$

$= \sqrt{2 - 2\cos t} = \sqrt{2}\sqrt{1-\cos t}$

can simplify using $1 - \cos t = 2\sin^2 \frac{t}{2}$ $\leadsto$ $|\vec{r}\,'(t)| = 2\left|\sin \frac{t}{2}\right|$

(b) $\boxed{\text{2 marks}}$ Find $\kappa(t)$, the curvature of the cycloid as a function of time t. What are the largest and smallest values of the curvature and where do they occur?

$K(t) = \dfrac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|^3}$

$\vec{r}\,'' = \langle \sin t, \cos t \rangle$

$\vec{r}\,' = \langle 1 - \cos t, \sin t \rangle$

$\vec{r}\,'' \times \vec{r}\,' = \langle 0, 0, \sin^2 t - \cos t(1 - \cos t) \rangle$

$= \langle 0, 0, 1 - \cos t \rangle$

$|\vec{r}\,'' \times \vec{r}\,'| = 1 - \cos t$ ← never negative

$K(t) = \dfrac{1 - \cos t}{2\sqrt{2}\,(1-\cos t)\sqrt{1-\cos t}}$

$\boxed{K(t) = \dfrac{1}{2\sqrt{2}\sqrt{1-\cos t}}}$

$K(t) = \infty$ max for $t = \ldots -2\pi, 0, 2\pi, 4\pi \ldots$

$K(t) = \frac{1}{4}$ min for $t = -3\pi, -\pi, \pi, 3\pi, \ldots$

K max when $\sqrt{1-\cos t}$ is smallest

K min " $\sqrt{1-\cos t}$ is biggest

can also write $K(t) = \dfrac{1}{4\left|\sin \frac{t}{2}\right|}$

(c) $\boxed{\text{2 marks}}$ Find the arclength of the part of the curve from $(0,0)$ to $(2\pi, 0)$. You may use the identity $1 - \cos(t) = 2\sin^2\left(\frac{t}{2}\right)$.

$|\vec{r}\,'(t)| = \sqrt{2}\sqrt{1-\cos t}$ using identity $|\vec{r}\,'(t)| = \sqrt{2}\sqrt{2\sin^2(\frac{t}{2})} = 2\left|\sin(\frac{t}{2})\right| = 2\sin\frac{t}{2}$ for $0 \le t \le 2\pi$ $\sin\frac{t}{2} \ge 0$

$$L = \int_{t=0}^{2\pi} 2\sin\frac{t}{2}\,dt = -4\cos\frac{t}{2}\Big]_0^{2\pi} = -4\cos\pi + 4\cos(0) = 4+4 = 8$$

(d) $\boxed{\text{2 marks}}$ Parameterize the part of the curve from $(0,0)$ to $(2\pi, 0)$ by arclength.

$$s(t) = \int_{u=0}^{t} 2\sin\frac{u}{2}\,du = -4\cos\frac{u}{2}\Big]_0^{t} = -4\cos\frac{t}{2} + 4$$

so $\frac{s}{4} = -\cos\frac{t}{2} + 1$

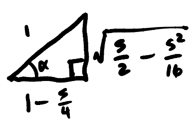
$\cos\frac{t}{2} = 1 - \frac{s}{4}$

$t = 2\cos^{-1}\left(1 - \frac{s}{4}\right)$

then $\vec{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ $0 \le t \le 2\pi$

$$\boxed{\vec{r}(s) = \left\langle 2\cos^{-1}\left(1-\frac{s}{4}\right) - \sin\left(2\cos^{-1}\left(1-\frac{s}{4}\right)\right), 1 - \cos\left(2\cos^{-1}\left(1-\frac{s}{4}\right)\right)\right\rangle \quad 0 \le s \le 8}$$

possible simplifications: let $\alpha = \cos^{-1}\left(1 - \frac{s}{4}\right)$ so



then $\sin(2\alpha) = 2\sin\alpha\cos\alpha$

$= 2\sqrt{\frac{s}{2} - \frac{s^2}{16}}\left(1 - \frac{s}{4}\right)$

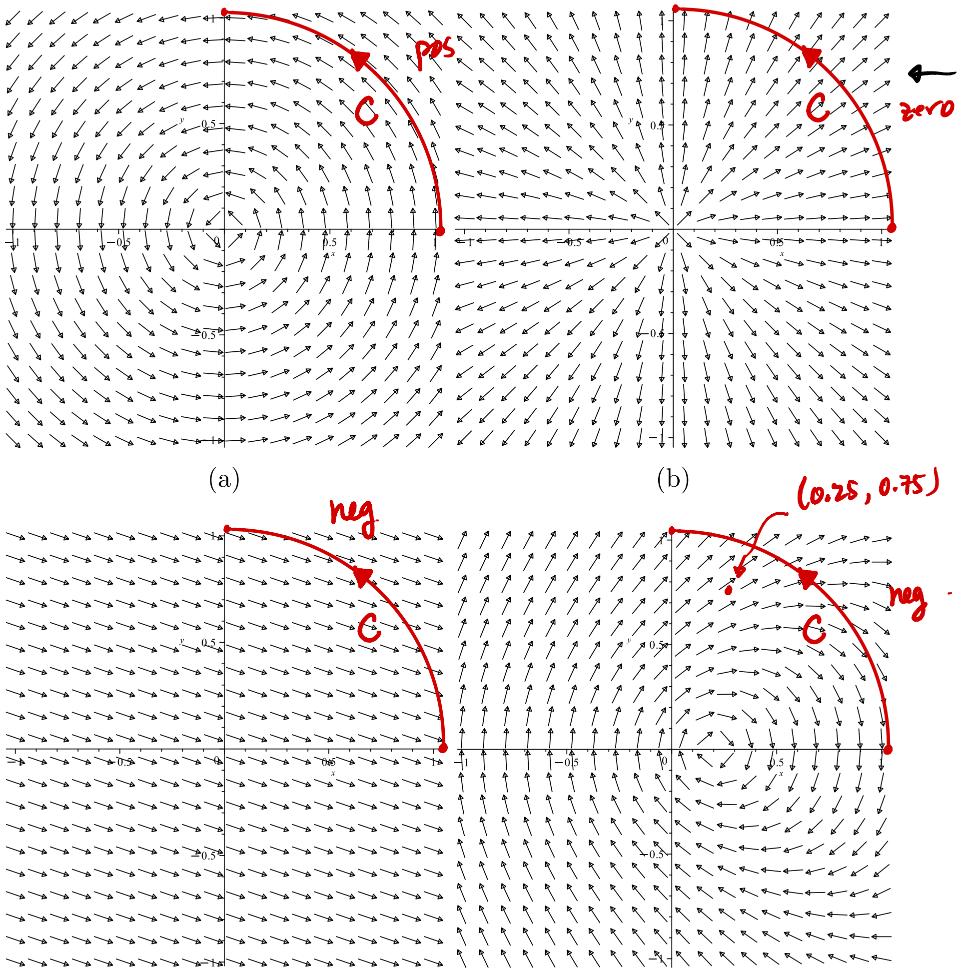
$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

$= \left(1 - \frac{s}{4}\right)^2 - \left(\frac{s}{2} - \frac{s^2}{16}\right)$

$= 1 - s + \frac{s^2}{8}$

$$\vec{r}(s) = \left\langle 2\cos^{-1}\left(1 - \frac{s}{4}\right) - 2\left(1 - \frac{s}{4}\right)\sqrt{\frac{s}{2} - \frac{s^2}{16}}, s - \frac{s^2}{8} \right\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$$



(a) (b) (c) (d)

$\vec{F}_b = \frac{\vec{F}}{|\vec{F}|} = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$

$\vec{F}_c$ is a const vector

2. The above plots depict four vector fields, $\vec{F}_a$, $\vec{F}_b$, $\vec{F}_c$, $\vec{F}_d$, in the xy-plane, plotted for $-1 \le x \le 1$ and $-1 \le y \le 1$. **Each of the vector fields has a constant magnitude.** You may assume that each vector field is continuous and differentiable on its domain.

   (a) $\boxed{\text{3 marks}}$ List all of the vector fields which satisfy the path-independence property.

   $\vec{F}_b$ & $\vec{F}_c$ ($\vec{F}_a$ and $\vec{F}_d$ have loops with non-zero work so don't satisfy path independence. $\vec{F}_b$ & $\vec{F}_c$ are conservative: $\vec{F}_c$ is curl free on a simply connected domain, $\vec{F}_b$ has potential function $f = \sqrt{x^2+y^2}$)

   (b) $\boxed{\text{3 marks}}$ List all of the vector fields which have a simply connected domain.

   only $\vec{F}_c$ (domain of $\vec{F}_a$ and $\vec{F}_b$ is $\mathbb{R}^2 - (0,0)$, domain of $\vec{F}_d$ is $\mathbb{R}^2 - \{(0.2,0)\}$)

(c) $\boxed{\text{4 marks}}$ Let C be the path from $(1,0)$ to $(0,1)$ given by the part of the unit circle with x and y positive. Say whether the following quantities are positive, negative, or zero:

1. $\int_C \vec{F}_a \cdot d\vec{r}$ positive
2. $\int_C \vec{F}_b \cdot d\vec{r}$ zero
3. $\int_C \vec{F}_c \cdot d\vec{r}$ negative
4. $\int_C \vec{F}_d \cdot d\vec{r}$ negative

(d) $\boxed{\text{4 marks}}$ Writing $\vec{F}_d = \langle P(x,y), Q(x,y)\rangle$, say whether the following quatities are positive, negative, or zero:

1. $P_x(0.25, 0.75)$ positive ($\vec{i}$ component of vector increases as x increases)
2. $P_y(0.25, 0.75)$ negative ($\vec{i}$ component of vector decreases as y increases)
3. $Q_x(0.25, 0.75)$ negative ($\vec{j}$ component of vector decreases as x increases)
4. $Q_y(0.25, 0.75)$ positive ($\vec{j}$ component of vector increases as y increases)

3. $\boxed{\text{6 marks}}$ Compute the work integral

$$\int_C (y\sin x + xy\cos x)dx + (x\sin x)dy$$

where C is any path from $\left(\frac{\pi}{4}, 1\right)$ to $\left(\frac{\pi}{2}, 2\right)$.

Integral can be written as $\int_C \vec{F}\cdot d\vec{r}$ where

$$\vec{F} = \langle P, Q\rangle = \langle y\sin x + xy\cos x,\ x\sin x\rangle$$

Domain of $\vec{F}$ is $\mathbb{R}^2$ so

$\vec{F}$ is conservative $\Leftrightarrow$ $P_y = Q_x$ which we check

$P_y = \sin x + x\cos x$

$Q_x = \sin x + x\cos x$

we don't require students to do this check since formulation of the question suggest $\vec{F}$ must be conservative.

Find potential function: $f_x = y\sin x + xy\cos x$ ①

$f_y = x\sin x$ ②

Taking antiderivative in equation ① requires integration by parts so it is easier to start with ②

② $\Rightarrow$ $f(x,y) = xy\sin x + h(x)$ $\Rightarrow$ $f_x = y\sin x + xy\cos x + h'(x) \underset{\text{by ①}}{=} y\sin x + xy\cos x$

so $h'(x) = 0 \Rightarrow h(x) = C$ constant

$f(x,y) = xy\sin x + C$ we can take $C = 0$

F.T.L.I $\Rightarrow$ $\int_C \vec{F}\cdot d\vec{r} = f\left(\frac{\pi}{2}, 2\right) - f\left(\frac{\pi}{4}, 1\right)$

$= \left(\frac{\pi}{2}\right)(2)\sin\frac{\pi}{2} - \left(\frac{\pi}{4}\right)(1)\sin\left(\frac{\pi}{4}\right)$

$= \pi - \frac{\pi}{4}\frac{1}{\sqrt{2}} = \pi\left(1 - \frac{\sqrt{2}}{8}\right)$

not defined for $y = \dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

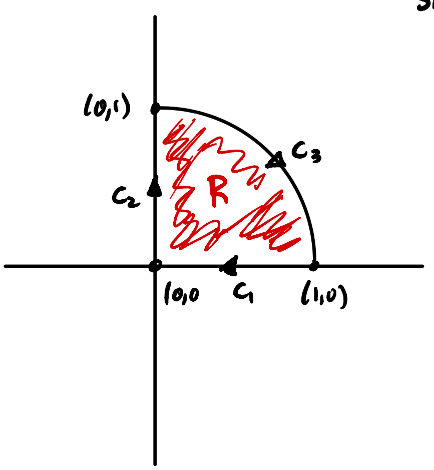
this makes part (b) of this problem badly formulated. this should have been something else like $\sin(y)$.

4. Consider the vector field

$$\vec{F} = \langle \underbrace{e^x - 4y + yx^2}_{P}, \underbrace{\tan(y) - xy^2}_{Q} \rangle$$

and let $C = C_1 + C_2 + C_3$ be the loop where C_1 is the line segment from $(1, 0)$ to $(0, 0)$, C_2 is the line segment from $(0, 0)$ to $(0, 1)$, and C_3 is the part of the circle $x^2 + y^2 = 1$ in the first quadrant from $(0, 1)$ to $(1, 0)$.

(a) $\boxed{\text{4 marks}}$ Compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$.



Since orientation on C is clockwise, we have $\partial R = -C$ and Green's theorem says

$$\oint_C \vec{F} \cdot d\vec{r} = -\oint_{-C} \vec{F} \cdot d\vec{r} = -\iint_R (Q_x - P_y)\,dx\,dy$$

$Q_x = -y^2 \quad P_y = -4 + x^2 \quad$ so $\quad Q_x - P_y = 4 - x^2 - y^2$

$$\oint_C \vec{F} \cdot d\vec{r} = -\iint_R (4 - x^2 - y^2)\,dx\,dy \quad \text{use polar coords}$$

$$= -\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} (4 - r^2)\, r\, dr\, d\theta$$

$$= \frac{\pi}{2} \int_{r=0}^{1} (r^3 - 4r)\,dr$$

$$= \frac{\pi}{2} \left[\frac{1}{4} r^4 - 2r^2 \right]_{r=0}^{1} = 2\pi\left(\frac{1}{4} - 2\right)$$

$$= \frac{\pi}{2}\left(-\frac{7}{4}\right) = -\frac{7\pi}{8}$$

(b) $\boxed{\text{4 marks}}$ Find a smooth, simple, closed, counterclockwise oriented curve $C_{\max}$, in the xy-plane for the which the value of the line integral

$$\oint_{C_{\max}} \vec{F} \cdot d\vec{r}$$

is a maximum among all smooth, simple, closed, counterclockwise oriented curves. Here $\vec{F}$ is as in part (a).

for any simple, closed, counterclockwise oriented curve C we have $C = \partial R$ where R is the region inside of C. We then want

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (Q_x - P_y)\,dx\,dy = \iint_R (4 - x^2 - y^2)\,dx\,dy$$ to be as big as possible.

(*)

To make the integral as big as possible we include in R everywhere where $4-x^2-y^2$ is positive and exclude everywhere $4-x^2-y^2$ is negative. Thus $R = \{4 - x^2 - y^2 \geq 0\} = \{x^2+y^2 \leq 4\}$

Thus R is the disk of radius 2 centered at the origin and so $C_{max} = \partial R$ is the circle of radius 2 centered at the origin.

$$C_{max} = \{x^2+y^2=4\}$$

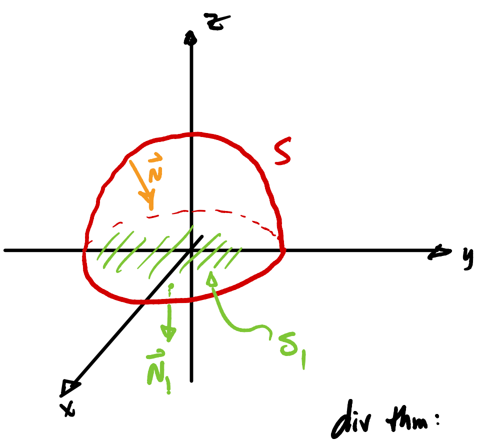
(*) this argument uses Green's theorem which requires $\vec{F}$ to be well defined on the domain of R which it won't be in general. This is a mistake in the formulation of this problem. I meant for $\vec{F}$ to be defined everywhere.

5. $\boxed{\text{8 marks}}$ Let S be the part of the surface $z = 1 - x^2 - y^2$ above the xy-plane, oriented downward. Let

$$\vec{F} = \left\langle ze^y + x^3, \tan(z) + y^3, 5 - 3z \right\rangle.$$

Compute the flux integral $\iint_S \vec{F} \cdot \hat{N}\, dS$. Hint: find a way to apply the divergence theorem.

$\operatorname{div} \vec{F} = 3x^2 + 3y^2 - 3$ ← nice, we would like to use div theorem



Let S_1 be the unit disk in the xy-plane $S_1 = \{z=0,\ x^2+y^2 \le 1\}$

oriented downward: $\vec{N}_1 = -\vec{k}$.

Let E be the solid region below S and above the xy-plane

$E = \{0 \le z \le 1 - x^2 - y^2\}$

Then $\partial E = S_1 - S$

minus since induced orientation on ∂E is outward, but given orientation on S is inward.

div thm:

$$\iiint_E 3(x^2+y^2-1)\, dVol = \iint_{S_1} \vec{F}\cdot\vec{N}\, dS - \underbrace{\iint_S \vec{F}\cdot\vec{N}\, dS}_{\text{we want this}}$$

$$\iint_S \vec{F}\cdot\vec{N}\, dS = \iint_{S_1} \vec{F}\cdot\vec{N}\, dS - \iiint_E 3(x^2+y^2-1)\, dVol$$

$$= \iint_{S_1} \langle \cdots, \cdots, 5 - 3\overset{0 \text{ since } z=0 \text{ on } S_1}{\cancel{z}} \rangle \cdot \langle 0,0,-1 \rangle\, dS - \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{1-r^2} 3(r^2-1)\, r\, dz\, dr\, d\theta$$

$$= -5 \iint_{S_1} dS - 2\pi \int_{r=0}^{1} 3r(r^2-1) \Big[z \Big]_0^{1-r^2} dr$$

$$= -5\, \text{Area}(S_1) - 2\pi \int_{r=0}^{1} -3r(r^2-1)^2\, dr$$

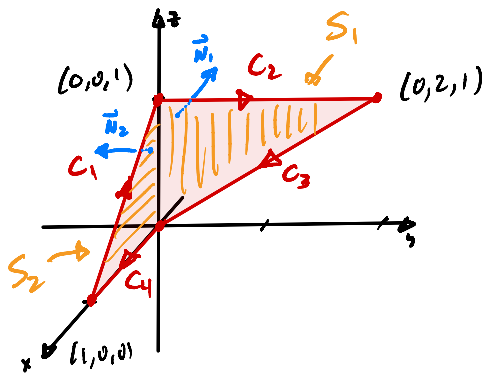
$$= -5\pi + 6\pi \left[\frac{1}{3\cdot 2} (r^2-1)^3 \right]_{r=0}^{1}$$

$$= -5\pi + \pi\left[(1-1)^3 - (-1)^3 \right] = -5\pi + \pi = -4\pi$$

6. $\boxed{\text{8 marks}}$ Let $C = C_1 + C_2 + C_3 + C_4$ be the loop consisting of 4 line segments: C_1 is the segment from $(1,0,0)$ to $(0,0,1)$, C_2 is the segment from $(0,0,1)$ to $(0,2,1)$, C_3 is the segment from $(0,2,1)$ to $(0,0,0)$, and C_4 is the segment from $(0,0,0)$ to $(1,0,0)$. Let $\vec{F}$ be the vector field

$$\vec{F} = \left\langle e^{x^2} + 3z - 6xyz, -\cos x - 3x^2 z, \sqrt{1+z^2} - 7y \right\rangle.$$

Compute the line integral $\oint_C \vec{F} \cdot d\vec{r}$.



$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{x^2}+3z-6xyz & -\cos x - 3x^2 z & \sqrt{1+z^2} - 7y \end{vmatrix}$$

$$= \langle -7 + 3x^2,\ 3 - 6xy,\ \sin x - \cancel{6xz} + \cancel{6xz} \rangle$$

Stoke's Theorem: $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ where $S = S_1 + S_2$ and S_1 is the triangular region in the yz plane with vertices $(0,0,1), (0,2,1), (0,0,0)$ oriented by $\vec{N}_1 = -\vec{i}$

S_2 is the triangular region in the xz plane with vertices $(1,0,0), (0,0,1), (0,0,0)$ oriented by $\vec{N}_2 = -\vec{j}$

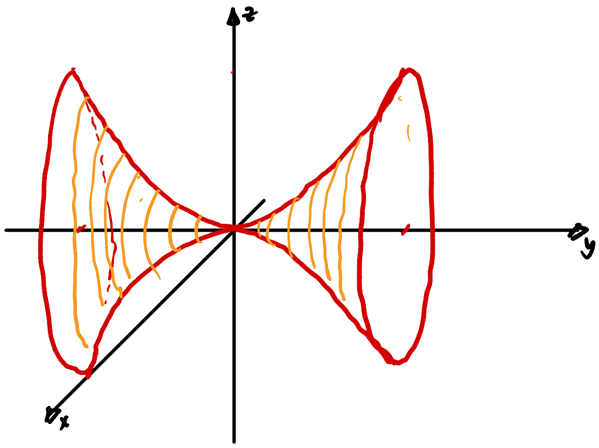
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_{S_1} \langle -7 + \overset{0 \text{ since } x=0}{\cancel{3x^2}}, \cdots, \cdots \rangle \cdot \langle -1,0,0\rangle \, dy\,dz + \iint_{S_2} \langle \cdots, 3 - \overset{0 \text{ since } y=0}{\cancel{6xy}}, \cdots \rangle \cdot \langle 0,-1,0 \rangle \, dx\,dz$$

$$= 7\,\text{Area}(S_1) - 3\,\text{Area}(S_2)$$

$$= 7 - \frac{3}{2} = \frac{11}{2}$$

7. Let S be the part of the surface $x^2+z^2=y^4$ lying between the planes $y=-1$ and $y=1$. Note that the surface S is the surface of revolution obtained by rotating the curve $\{x=0, z=y^2, -1 \le y \le 1\}$ about the y-axis.

(a) $\boxed{\text{3 marks}}$ Sketch the surface S and find a parameterization of S (be sure to include the domain as well as the vector valued function).



for each fixed y, $x^2+z^2=y^4$ is a circle of radius y^2. We can use parameters y & θ

$$\boxed{\vec{r}(y,\theta) = \langle y^2\cos\theta,\ y,\ y^2\sin\theta \rangle \qquad \begin{array}{l} -1 \le y \le 1 \\ 0 \le \theta \le 2\pi \end{array}}$$

as a check, we can see that the parameterization satisfies the equation:

$$(y^2\cos\theta)^2 + (y^2\sin\theta)^2 = y^4$$

$$y^4(\cos^2\theta + \sin^2\theta) = y^4 \quad \checkmark$$

(b) $\boxed{\text{3 marks}}$ Compute the integral $\iint_S \frac{1}{\sqrt{1+4y^2}} dS$.

$$dS = |\vec{r}_y \times \vec{r}_\theta|\, dy\, d\theta$$

$$\vec{r}_y = \langle 2y\cos\theta,\ 1,\ 2y\sin\theta \rangle$$

$$\vec{r}_\theta = \langle -y^2\sin\theta,\ 0,\ y^2\cos\theta \rangle$$

$$\vec{r}_y \times \vec{r}_\theta = \langle y^2\cos\theta,\ -2y^3\sin^2\theta - 2y^3\cos^2\theta,\ y^2\sin\theta \rangle$$

$$= y^2 \langle \cos\theta,\ -2y,\ \sin\theta \rangle$$

$$|\vec{r}_y \times \vec{r}_\theta| = y^2 |\langle \cos\theta, -2y, \sin\theta \rangle|$$

$$= y^2 \sqrt{\cos^2\theta + 4y^2 + \sin^2\theta}$$

$$= y^2\sqrt{1+4y^2}$$

so

$$\iint_S \frac{1}{\sqrt{1+4y^2}} dS = \int_{\theta=0}^{2\pi} \int_{y=-1}^{1} \frac{y^2\sqrt{1+4y^2}}{\sqrt{1+4y^2}}\, dy\, d\theta$$

$$= 2\pi \int_{y=-1}^{1} y^2\, dy = 2\pi \left[\frac{y^3}{3}\right]_{-1}^{1}$$

$$= 2\pi\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{4\pi}{3}$$

8. The paraboloidal coordinate system is an unusual coordinate system on $\mathbb{R}^3$ which is very useful in certain circumstances. The coordinates are (u, v, θ) and the standard Cartesian coordinates can be expressed in terms of the paraboloidal coordinates by the equations

$$x = (u+v)\cos\theta$$
$$y = (u+v)\sin\theta$$
$$z = u - v$$

(a) $\boxed{\text{1 mark}}$ Express dx, dy, and dz in terms of du, dv, and $d\theta$.

$dx = \cos\theta\, du + \cos\theta\, dv - (u+v)\sin\theta\, d\theta$

$dy = \sin\theta\, du + \sin\theta\, dv + (u+v)\cos\theta\, d\theta$

$dz = du - dv$

(b) $\boxed{\text{2 marks}}$ Express the Cartesian volume form $dx \wedge dy \wedge dz$ in terms of the paraboloidal volume form $du \wedge dv \wedge d\theta$. Simplify as much as possible (your answer should be quite simple).

In general, $(A_1 du + A_2 dv + A_3 d\theta) \wedge (B_1 du + B_2 dv + B_3 d\theta) \wedge (C_1 du + C_2 dv + C_3 d\theta)$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} du \wedge dv \wedge d\theta$$

determinant

so $dx \wedge dy \wedge dz = \begin{vmatrix} \cos\theta & \cos\theta & -(u+v)\sin\theta \\ \sin\theta & \sin\theta & (u+v)\cos\theta \\ 1 & -1 & 0 \end{vmatrix} du \wedge dv \wedge d\theta$

$$= \left(\cos\theta \begin{vmatrix} \sin\theta & (u+v)\cos\theta \\ -1 & 0 \end{vmatrix} - \cos\theta \begin{vmatrix} \sin\theta & (u+v)\cos\theta \\ 1 & 0 \end{vmatrix} - (u+v)\sin\theta \begin{vmatrix} \sin\theta & \sin\theta \\ 1 & -1 \end{vmatrix} \right) du \wedge dv \wedge d\theta$$

$$= \left[\cos^2\theta\,(u+v) + \cos^2\theta\,(u+v) - (u+v)\sin\theta\,(-2\sin\theta)\right] du \wedge dv \wedge d\theta$$

$$= \left[2(u+v)(\cos^2\theta + \sin^2\theta)\right] du \wedge dv \wedge d\theta = 2(u+v)\, du \wedge dv \wedge d\theta$$