Mathematics 317 — Midterm 1 — 50 minutes

February 12, 2025

- The test consists of 9 pages and 4 questions worth a total of 34 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, formula sheets other than the one provided, or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Signature	Solutions							
Name								

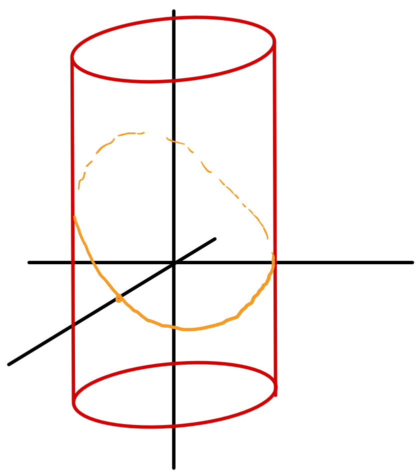
Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. Let C be the curve given by the intersecction of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$.

   (a) $\boxed{\text{3 marks}}$ Find a parameterization of C (be sure to include the domain as well as the vector valued function).



$$\vec{r}(t) = \langle \underbrace{\cos t, \sin t}_{\text{projection of } C \text{ to } xy \text{ plane is circle}}, \underbrace{1 - \cos t - \sin t}_{\text{need } z = 1 - x - y} \rangle \qquad 0 \le t \le 2\pi$$

   (b) $\boxed{\text{3 marks}}$ Compute the integral $\int_C f\, ds$ where $f(x, y, z) = \sqrt{x + y + z - xy}$.

$$\vec{r}'(t) = \langle -\sin t, \cos t, \sin t - \cos t \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + (\sin t - \cos t)^2} = \sqrt{1 + \sin^2 t - 2\sin t\cos t + \cos^2 t}$$

$$= \sqrt{2 - 2\sin t\cos t}$$

$$\int_C f\, ds = \int_{t=0}^{2\pi} f(\vec{r}(t))\, |\vec{r}'(t)|\, dt$$

$$= \int_{t=0}^{2\pi} \sqrt{1 - \cos t \sin t}\, \sqrt{2 - 2\sin t\cos t}\, dt$$

$$= \int_{t=0}^{2\pi} \sqrt{2}(1 - \cos t \sin t)\, dt = \left[\sqrt{2}t - \frac{\sqrt{2}}{2}\sin^2 t\right]_0^{2\pi} = \sqrt{2}\cdot 2\pi$$

2. $\boxed{\text{9 marks}}$ Match the following vector valued functions to the following plots. The axes of the plots have $-2 \le x, y \le 2$.

d (1) $\left\langle e^{t/8}\cos(t), e^{t/8}\sin(t), \right\rangle, \quad -2\pi \le t \le 2\pi$ goes around origin twice with distance to origin growing

b (2) $\langle 2t^2, 2t^3 \rangle, \quad -1 \le t \le 1$ x is positive, y is both neg & positive ⇒ curve is in 1st and 4th quadrants

f (3) $\langle 2t\,e^{t}, 2t\,e^{-t} \rangle, \quad -1 \le t \le 1$ x,y are both negative or both positive ⇒ lives in 1st and 3rd quadrants

h (4) $\langle 2\sin(2\pi t), 4t^2 - 2 \rangle, \quad -1 \le t \le 1$ when $t \leftrightarrow -t$ $(x,y) \leftrightarrow (-x,y)$ ⇒ curve symmetric under reflection about y-axis

a (5) $\langle 6t^3 + 3t, -2t^3 - t - 1 \rangle, \quad -1 \le t \le 1$ satisfies $2y + x = -2$

g (6) $\langle e^{t} - 1, e^{-t} - 1 \rangle, \quad -2 \le t \le 2$ satisfies $(x+1)(y+1) = 1$ (so like the hyperbola $xy = 1$ except shifted by $\langle -1, -1 \rangle$)

i (7) $\left\langle \frac{4t}{t^2+1}, \frac{2(1-t^2)}{t^2+1} \right\rangle, \quad -\infty \le t \le \infty$ satisfies $x^2 + y^2 = 4$

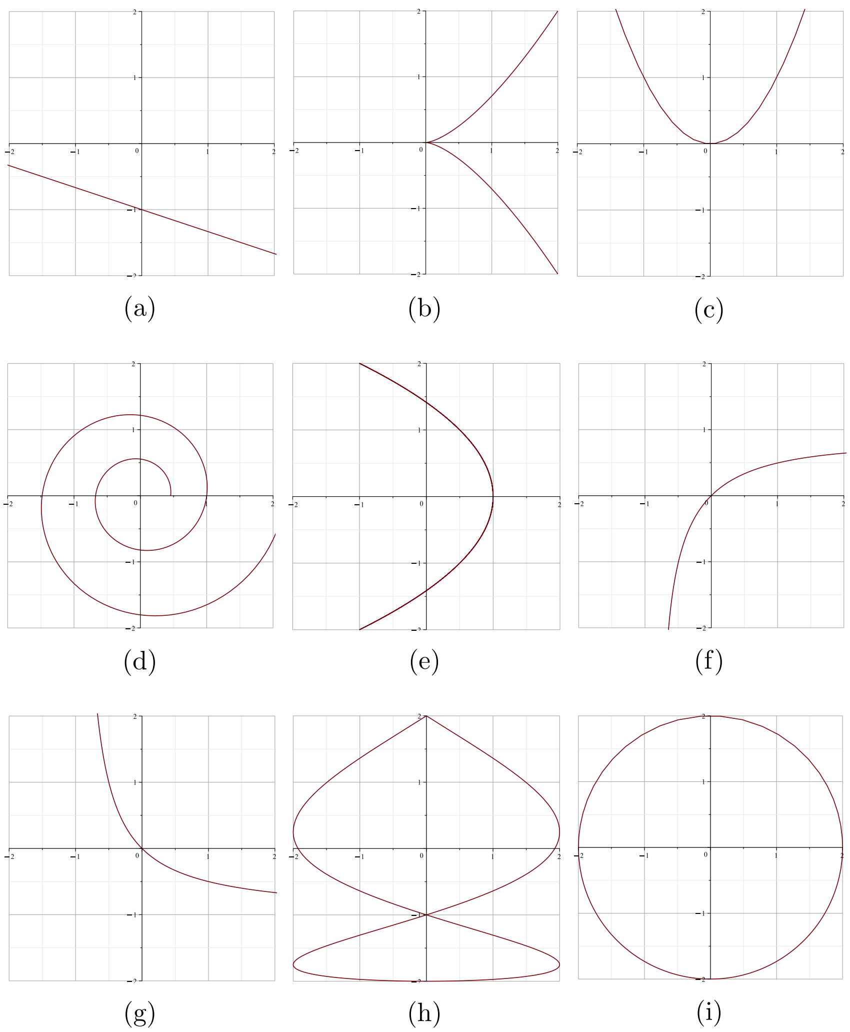
c (8) $\langle e^{t} - e^{-t}, e^{2t} + e^{-2t} - 2 \rangle, \quad -\infty \le t \le \infty$ satisfies $y = x^2$

e (9) $\langle \cos(2\pi t), 2\sin(\pi t) \rangle, \quad -2 \le t \le 2$ $\langle \cos(2\pi t), 2\sin(\pi t) \rangle = \langle \cos^2(\pi t) - \sin^2(\pi t), 2\sin(\pi t) \rangle$

$= \langle 1 - 2\sin^2(\pi t), 2\sin(\pi t) \rangle$

so satisfies

$x = 1 - 2y^2$



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

3. Consider the parameterized curve

$$\vec{r}(t) = \left\langle 1 - 2t^2, 2t\sqrt{1-t^2} \right\rangle, \quad 0 \le t \le 1.$$

(a) $\boxed{\text{3 marks}}$ Find and simplify $|\vec{r}'(t)|$.

$$\vec{r}\,' = \left\langle -4t,\ 2\sqrt{1-t^2} + 2t \cdot \frac{-t}{\sqrt{1-t^2}} \right\rangle = 2\left\langle -2t, \frac{(1-t^2) - t^2}{\sqrt{1-t^2}} \right\rangle$$

$$|\vec{r}\,'| = 2\sqrt{4t^2 + \frac{(1-2t^2)^2}{1-t^2}}$$

$$= 2\sqrt{\frac{4t^2(1-t^2) + (1-4t^2+4t^4)}{1-t^2}}$$

$$= \frac{2}{\sqrt{1-t^2}} \cdot \sqrt{\cancel{4t^2} - \cancel{4t^4} + 1 - \cancel{4t^2} + \cancel{4t^4}} = \frac{2}{\sqrt{1-t^2}}$$

(b) $\boxed{\text{3 marks}}$ Find the distance from $t = 0$ to $t = T$ along the curve.

$$S = \int_0^T |\vec{r}\,'(t)|\,dt = \int_{t=0}^{T} \frac{2}{\sqrt{1-t^2}}\,dt$$

let $t = \sin u$, $dt = \cos u\,du$

$t = 0 \Leftrightarrow u = 0$

$t = T \Leftrightarrow u = \sin^{-1} T$

$$S = \int_{u=0}^{\sin^{-1}T} \frac{2\cos u\,du}{\sqrt{1-\sin^2 u}} = \int_{u=0}^{\sin^{-1}T} 2\frac{\cos u}{|\cos u|}\,du$$

since $0 \le t \le 1 \Rightarrow 0 \le u \le \pi/2 \Rightarrow \cos u \ge 0 \Rightarrow |\cos u| = \cos u$

$$= \int_{u=0}^{\sin^{-1}T} 2\,du = 2\sin^{-1} T$$

(c) $\boxed{\text{3 marks}}$ Reparameterize the curve by arclength.

$s(t) = 2\sin^{-1}(t)$ so $t = \sin\frac{s}{2}$ $\quad 0 \le t \le 1 \iff 0 \le s \le \pi$

$\vec{r}(s) = \langle 1 - 2\sin^2\frac{s}{2}, 2\sin\frac{s}{2}\sqrt{1-\sin^2\frac{s}{2}} \rangle = \langle 1 - 2\sin^2\frac{s}{2}, 2\sin\frac{s}{2}\cos\frac{s}{2} \rangle \quad 0 \le s \le \pi$

$= \langle \cos^2\frac{s}{2} - \sin^2\frac{s}{2}, 2\sin\frac{s}{2}\cos\frac{s}{2} \rangle \quad 0 \le s \le \pi$

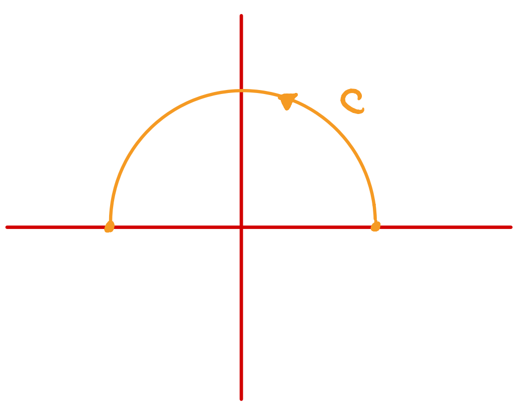
(d) $\boxed{\text{2 marks}}$ Geometrically describe the curve.

$\vec{r}(s) = \langle \cos^2\frac{s}{2} - \sin^2\frac{s}{2}, 2\sin\frac{s}{2}\cos\frac{s}{2} \rangle \quad 0 \le s \le \pi$ $\quad$ Using $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= \langle \cos s, \sin s \rangle \quad 0 \le s \le \pi$

so this is the top half of the unit circle $\{x^2+y^2=1 \;\; y \ge 0\}$



4. Consider a racetrack given by the logarithmic spiral

$$\vec{r}(\theta) = \left\langle e^{\theta}\cos\theta, e^{\theta}\sin\theta \right\rangle, \quad 0 \le \theta \le 2\pi.$$

(a) $\boxed{\text{4 marks}}$ Compute the curvature $\kappa(\theta)$ as a function of the angle θ. Simplify as much as possible.

$$\vec{r}\,'(\theta) = \langle e^{\theta}\cos\theta - e^{\theta}\sin\theta,\ e^{\theta}\sin\theta + e^{\theta}\cos\theta \rangle$$

$$= e^{\theta}\langle \cos\theta - \sin\theta,\ \cos\theta + \sin\theta \rangle$$

$$|\vec{r}\,'(\theta)| = e^{\theta}\sqrt{(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2}$$

$$= e^{\theta}\sqrt{2\cos^2\theta + 2\sin^2\theta} = e^{\theta}\sqrt{2}$$

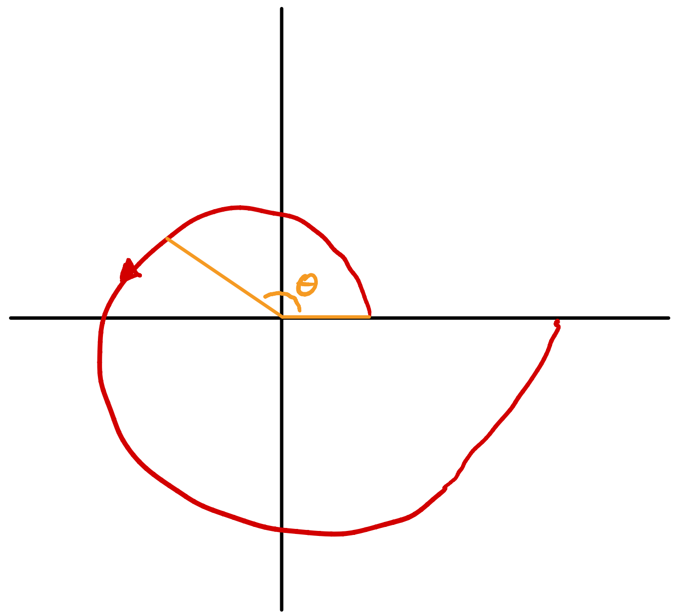
so $\vec{T}(\theta) = \frac{1}{\sqrt{2}}\langle \cos\theta - \sin\theta,\ \cos\theta + \sin\theta \rangle$

$$\vec{T}\,'(\theta) = \frac{1}{\sqrt{2}}\langle -\sin\theta - \cos\theta,\ -\sin\theta + \cos\theta \rangle$$

$$|\vec{T}\,'(\theta)| = \frac{1}{\sqrt{2}}\sqrt{(-\sin\theta - \cos\theta)^2 + (-\sin\theta + \cos\theta)^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 2\cos^2\theta} = 1$$

$$\kappa(\theta) = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{1}{\sqrt{2}\,e^{\theta}}$$



(b) $\boxed{\text{4 marks}}$ The friction between the road and the tires of a racecar is what keeps it from skidding around turns. The friction is such that a racecar of mass m can withstand a force in the principle normal direction of up to and including μm without skidding (μ is the coefficient of friction).

Find $\theta(t)$, the angle as a function of time, so that a racecar travelling with position vector $\vec{r}(\theta(t))$ is travelling as fast as possible at all times without skidding. Hint: the component of the force in the normal direction of travel should be exactly μm. Assume that at $t = 0$, $\theta = 0$.

Force in $\vec{N}$ direction is $m a_N = m \kappa \left|\frac{d\vec{r}}{dt}\right|^2$ and we want it equal to μm

$$\Rightarrow \quad \cancel{m} \kappa \left|\frac{d\vec{r}}{dt}\right|^2 = \cancel{m}\mu$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{d\theta}\frac{d\theta}{dt} \quad \text{so} \quad \left|\frac{d\vec{r}}{dt}\right|^2 = \left|\frac{d\vec{r}}{d\theta}\right|^2 \left(\frac{d\theta}{dt}\right)^2 = 2e^{2\theta}\left(\frac{d\theta}{dt}\right)^2$$

$$\text{so} \quad \kappa\left|\frac{d\vec{r}}{dt}\right|^2 = \mu \iff \frac{1}{\sqrt{2}\,e^{\theta}} \cdot 2e^{2\theta}\left(\frac{d\theta}{dt}\right)^2 = \mu \iff \sqrt{2}\,e^{\theta}\left(\frac{d\theta}{dt}\right)^2 = \mu$$

$$2^{1/4} e^{\theta/2} \frac{d\theta}{dt} = \sqrt{\mu} \quad \Rightarrow \quad \int 2^{1/4} e^{\theta/2}\, d\theta = \int \sqrt{\mu}\, dt$$

positive since θ increases as t increases

$$\Rightarrow \quad 2^{1/4} \cdot 2e^{\theta/2} = \sqrt{\mu}\, t + C$$

when $t=0$ $\theta=0$ $\Rightarrow$ $C = 2^{1/4} \cdot 2 = 2^{5/4}$

$$\Rightarrow \quad 2^{5/4} e^{\theta/2} = \sqrt{\mu}\, t + 2^{5/4}$$

$$\Rightarrow \quad e^{\theta/2} = 2^{-5/4}\sqrt{\mu}\, t + 1$$

$$\Rightarrow \quad \theta(t) = 2\log\left(2^{-5/4}\sqrt{\mu}\, t + 1\right)$$

This blank page can be used for spill over work. If you continue work from another problem, be sure to indicate that on the original page.