Mathematics 317 — Midterm 2 — 50 minutes

March 14, 2025

- The test consists of 10 pages and 4 questions worth a total of 33 marks.
- This is a closed-book examination. You are allowed to bring in and use one formula sheet (this sheet should be reasonable: for example you shouldn't need special glasses or intsruments to read it). **None of the following are allowed**: documents, formula sheets other than the one you came with, or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Signature	Solutions							
Name								

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. For each statement below, say whether the statement is true or false. If it is true, give a reason why it is true (this reason could include "this is a theorem in the book" or "we proved this in class"). If it is false, supply a counterexample that demonstrates that the statement is false. You may assume that all vector fields in the follow statements are reasonable: they are continuous and that all derivatives of their components exist.

   (a) $\boxed{\text{2 marks}}$ Let $U \subset V \subset \mathbb{R}^2$ be open domains. If a vector field $\vec{F}$ is conservative on V, then $\vec{F}$ is conservative when restricted to U.

True : $\vec{F}$ conservative on V means there is a function f on V such that $\vec{\nabla} f = \vec{F}$ then f restricted to U is a potential function for $\vec{F}$ restrict to U

   (b) $\boxed{\text{2 marks}}$ Let $U \subset V \subset \mathbb{R}^2$ be open domains. If a vector field $\vec{F}$ is not conservative on V, then $\vec{F}$ is not conservative when restricted to U.

False : $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ is not conservative on $V = \mathbb{R}^2 - (0,0)$ but restricted to $U = \mathbb{R}^2 - \{\text{pos. } x\text{-axis}\}$ is conservative with potential function θ. (we discussed this in class).

(c) $\boxed{\text{2 marks}}$ Suppose the domain of $\vec{F}$ is $\mathbb{R}^3$ minus the z-axis. If curl $\vec{F} = \vec{0}$ then $\vec{F}$ is conservative.

False $\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$ is a counter example. it has curl $= \vec{0}$ but it is not conservative (domain is not simply connected).

(d) $\boxed{\text{2 marks}}$ Suppose the domain of $\vec{F}$ is $\mathbb{R}^3$ minus the z-axis. If $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all loops C, then $\vec{F}$ is conservative.

True: This is a theorem we proved in class. It holds for any domain.

(e) $\boxed{\text{2 marks}}$ Let S be the sphere $\{x^2 + y^2 + z^2 = 1\}$ oriented outward. Suppose that the flux integral $\iint_S \vec{F} \cdot d\vec{S} = 0$, then for all points (x, y, z) on the sphere $\vec{F}(x, y, z) \cdot \langle x, y, z \rangle = 0$.

False : $\vec{F} = \langle 0,0,1 \rangle$ is a counter example since $\vec{F} \cdot \vec{N}$ is not zero at the north pole and since $\vec{N} = \langle x,y,z \rangle$ $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 0,0,1\rangle \cdot \langle x,y,z\rangle dS = \iint_S z\,dS = 0$ by symmetry $z \leftrightarrow -z$.

(f) $\boxed{\text{2 marks}}$ Let P be the plane $\{x + y + z = 1\}$ oriented upward. Suppose that the flux integral $\iint_S \vec{F} \cdot d\vec{S} = 0$ for every closed and bounded region $S \subset P$, then for all points (x, y, z) on the plane $\vec{F}(x, y, z) \cdot \langle 1, 1, 1 \rangle = 0$.

True : $\vec{N} = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle$. Suppose for the sake of contradiction that $\vec{F}(x_0,y_0,z_0) \cdot \langle 1,1,1 \rangle \neq 0$ for some (x_0,y_0,z_0). Then by continuity, $\vec{F}(x,y,z) \cdot \langle 1,1,1\rangle \neq 0$ for all $(x,y,z) \in B_\varepsilon(x_0,y_0,z_0)$ where $B_\varepsilon(x_0,y_0,z_0)$ is the ball of radius ε centered at (x_0,y_0,z_0) and $\varepsilon > 0$ is sufficiently small. Then let $S \subset P$ be $B_\varepsilon(x_0,y_0,z_0) \cap P$ so $\iint_S \vec{F} \cdot \vec{N} dS = \frac{1}{\sqrt{3}} \iint_S \vec{F} \cdot \langle 1,1,1\rangle dS$ is non-zero since $\vec{F} \cdot \langle 1,1,1 \rangle$ is a non-zero continuous function on S. This contradicts the hypothesis so $\vec{F}(x_0,y_0,z_0) \cdot \langle 1,1,1\rangle = 0$ $\forall (x_0,y_0,z_0) \in P$.

2. (a) $\boxed{\text{4 marks}}$ Find the values of the constants A and B such the the vector field

$$\vec{F} = e^{2x+3y} \langle A + 2x + 2z, Bx + 3z, 1\rangle$$

is conservative.

Since the domain of $\vec{F}$ is $\mathbb{R}^3$ which is simply connected, $\vec{F}$ is conservative $\Leftrightarrow$ curl $\vec{F} = \vec{0}$

We can compute curl $\vec{F}$ directly or with curl$(f\vec{G}) = \vec{\nabla} f \times \vec{G} + f\,\vec{\nabla}\times\vec{G}$ with $f = e^{2x+3y}$ $\vec{G} = \langle A+2x+2z, Bx+3z, 1\rangle$

$\vec{\nabla} f = e^{2x+3y}\langle 2,3,0\rangle$ $\quad \vec{\nabla}\times\vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A+2x+2z & Bx+3z & 1 \end{vmatrix} = \langle -3, 2, B\rangle$

$$\vec{\nabla} f \times \vec{G} = e^{2x+3y}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ A+2x+2z & Bx+3z & 1 \end{vmatrix} = e^{2x+3y}\langle 3, -2, 2(Bx+3z) - 3(A+2x+2z)\rangle$$

$$= e^{2x+3y}\langle 3, -2, (2B-6)x + \cancel{6z} - 3A - \cancel{6z}\rangle$$

So $\vec{\nabla}\times\vec{F} = e^{2x+3y}\langle -3+3, 2-2, (B-3A) + (2B-6)x\rangle = e^{2x+3y}\langle 0, 0, (B-3A) + (2B-6)x\rangle$

so $B = 3A$ and

$2B = 6$

$\Rightarrow \boxed{B = 3 \quad A = 1}$

(b) $\boxed{\text{4 marks}}$ Recall that $\vec{F} = e^{2x+3y}\langle A + 2x + 2z, Bx + 3z, 1\rangle$. Using the values of A and B found in part (a), compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ where C is any curve starting at $(1, 1, 1)$ and ending at $(2, 2, 2)$

$A=1 \quad B=3 \quad \vec{F} = \langle e^{2x+3y}(1+2x+2z), e^{2x+3y}(3x+3z), e^{2x+3y}\rangle$

we want $f(x,y,z)$ with

① $f_x = e^{2x+3y}(1+2x+2z)$

② $f_y = e^{2x+3y}(3x+3z)$

③ $f_z = e^{2x+3y}$

③ $\Rightarrow f(x,y,z) = ze^{2x+3y} + g(x,y) \Rightarrow f_y = 3ze^{2x+3y} + g_y(x,y)$

then ② $\Rightarrow e^{2x+3y}(3x+\cancel{3z}) = \cancel{3ze^{2x+3y}} + g_y(x,y) \Rightarrow g_y(x,y) = 3xe^{2x+3y} \Rightarrow g(x,y) = xe^{2x+3y} + h(x)$

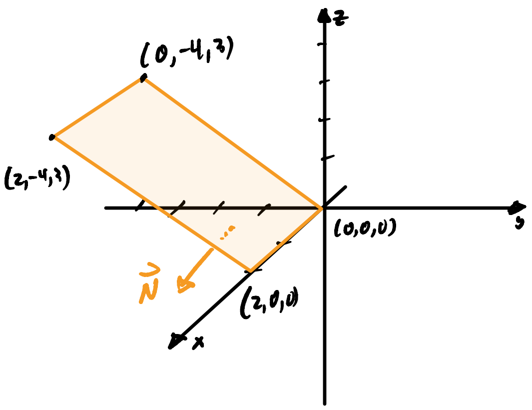
so $f(x,y,z) = ze^{2x+3y} + xe^{2x+3y} + h(x)$

$\Rightarrow f_x = 2ze^{2x+3y} + e^{2x+3y} + 2xe^{2x+3y} + h'(x) = (1+2x+2z)e^{2x+3y} + h'(x)$ so then ① $\Rightarrow h'(x)=0$ so $h(x)$ = constant which we may assume is 0.

$\Rightarrow f(x,y,z) = (x+z)e^{2x+3y}$

so $\int_C \vec{F}\cdot d\vec{r} = f(2,2,2) - f(1,1,1) = 4e^{10} - 2e^{5}$

3. $\boxed{\text{5 marks}}$ Let R be the rectangle in $\mathbb{R}^3$ with vertices at $(0,0,0)$, $(2,0,0)$, $(0,-4,3)$, and $(2,-4,3)$ oriented downward. Let $\vec{F} = \langle xye^z, 7-4xy, 6+3xy\rangle$. Compute the flux integral $\iint_S \vec{F}\cdot d\vec{S}$.



R has a constant Normal vector which we can get by $\langle 2,0,0\rangle \times \langle 0,-4,3\rangle$ (and then make a unit vector)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & -4 & 3 \end{vmatrix} = \langle 0,-6,-8\rangle \qquad |\langle 0,-6,-8\rangle| = \sqrt{36+64} = 10$$

so $\vec{N} = \langle 0, -\frac{3}{5}, -\frac{4}{5}\rangle$

$$\vec{F}\cdot\vec{N} = \langle xye^z, 7-4xy, 6+3xy\rangle \cdot \langle 0,-\tfrac{3}{5},-\tfrac{4}{5}\rangle$$
$$= 0 - \tfrac{3}{5}(7-4xy) - \tfrac{4}{5}(6+3xy)$$
$$= -\tfrac{21}{5} + \tfrac{12}{5}xy - \tfrac{24}{5} - \tfrac{12}{5}xy = -\tfrac{45}{5} = -9$$

so $\iint_R \vec{F}\cdot d\vec{S} = \iint_R \vec{F}\cdot\vec{N}\,dS = -9\iint_R dS = -9\,\text{Area}(R)$

R is a rectangle with side lengths 2 and $\sqrt{4^2+3^2} = 5$ so Area R = 10

$$\iint_R \vec{F}\cdot d\vec{S} = -90$$

Or parameterize R: $\vec{r}(x,y) = \langle x, y, -\frac{3}{4}y\rangle \quad 0\le x\le 2,\ -4\le y\le 0$

$\vec{r}_x = \langle 1,0,0\rangle$

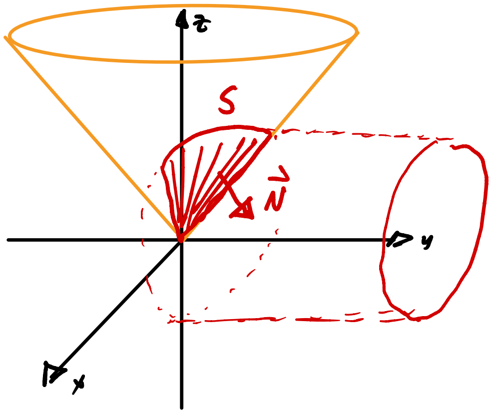
$\vec{r}_y = \langle 0,1,-\frac{3}{4}\rangle$

$\vec{r}_x\times\vec{r}_y = \langle 0,\frac{3}{4},1\rangle \longleftarrow$ opposite from $\vec{N}$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{x=0}^{2}\int_{y=-4}^{0} \langle xye^{-\frac{3}{4}y}, 7-4xy, 6+3xy\rangle\cdot\langle 0,-\tfrac{3}{4},-1\rangle\,dy\,dx$$
$$= \int_{x=0}^{2}\int_{y=-4}^{0}\left[-\tfrac{3}{4}(7-\cancel{4xy}) - (6+\cancel{3xy})\right]dy\,dx$$
$$= \left(-\tfrac{21}{4}-6\right)\int_{x=0}^{2}\int_{y=-4}^{0} dy\,dx = \left(-\tfrac{21}{4}-6\right)\cdot 8$$
$$= -42-48 = -90$$

4. Let S be the part of the cone $z = \sqrt{x^2+y^2}$ which is contained inside the cylinder $x^2+z^2=1$ and has $x \geq 0$ and $y \geq 0$. Assume that S is oriented in the downward direction.

(a) $\boxed{\text{4 marks}}$ Find a parameterization of S (be sure to include the domain as well as the vector valued function).



① $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ $x^2+z^2 \leq 1 \Rightarrow r^2\cos^2\theta + r^2 \leq 1 \Rightarrow r^2 \leq \frac{1}{1+\cos^2\theta}$

$r \leq \frac{1}{\sqrt{1+\cos^2\theta}}$

domain: $0 \leq \theta \leq \frac{\pi}{2} \leftarrow x,y \geq 0$

$0 \leq r \leq \frac{1}{\sqrt{1+\cos^2\theta}}$

or ② $\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$ $x^2+z^2 \leq 1 \Rightarrow x^2+x^2+y^2 \leq 1$

domain: $2x^2+y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

for use in the next part: $\vec{r}_r = \langle \cos\theta, \sin\theta, 1\rangle$

$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$

$\vec{r}_r \times \vec{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle$ opposite from $\vec{N}$

$\vec{r}_x = \langle 1, 0, \frac{x}{\sqrt{x^2+y^2}}\rangle$

$\vec{r}_y = \langle 0, 1, \frac{y}{\sqrt{x^2+y^2}}\rangle$

$\vec{r}_x \times \vec{r}_y = \langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\rangle$ opposite from $\vec{N}$

(b) $\boxed{\text{4 marks}}$ Compute the flux integral $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle zy, zx, 0\rangle$.

① $\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$

domain: $0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le \frac{1}{\sqrt{1+\cos^2\theta}}$

$\vec{r}_r \times \vec{r}_\theta = \langle -r\cos\theta, -r\sin\theta, r\rangle$ opposite from $\vec{N}$

$\vec{F} = \langle r^2\sin\theta, r^2\cos\theta, 0\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\frac{1}{\sqrt{1+\cos^2\theta}}} \langle r^2\sin\theta, r^2\cos\theta, 0\rangle \cdot \langle r\cos\theta, r\sin\theta, -r\rangle \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{\frac{1}{\sqrt{1+\cos^2\theta}}} 2r^3 \sin\theta\cos\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{2} r^4 \Big]_0^{\frac{1}{\sqrt{1+\cos^2\theta}}} \sin\theta\cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \frac{1}{2}\cdot\frac{\sin\theta\cos\theta}{(1+\cos^2\theta)^2} \, d\theta \qquad \begin{aligned} u &= 1+\cos^2\theta \\ du &= -2\cos\theta\sin\theta \, d\theta \end{aligned}$$

$$= \int_{u=2}^{1} -\frac{1}{4}\frac{du}{u^2} = \frac{1}{4}u^{-1}\Big]_2^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

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② $\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$

domain: $2x^2+y^2 \le 1$, $x \ge 0$, $y \ge 0$.

$\vec{r}_x \times \vec{r}_y = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle$ opposite from $\vec{N}$

$\vec{F} = \langle y\sqrt{x^2+y^2}, x\sqrt{x^2+y^2}, 0\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\left\{\substack{2x^2+y^2\le 1 \\ x\ge 0\ y \ge 0}\right\}} \langle y\sqrt{x^2+y^2}, x\sqrt{x^2+y^2}, 0\rangle \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 1\right\rangle dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{\sqrt{\frac{1-y^2}{2}}} 2xy \, dx\, dy$$

$$= \int_{y=0}^{1} \left[x^2 y\right]_0^{\sqrt{\frac{1-y^2}{2}}} dy = \int_{y=0}^{1} \frac{1}{2}(1-y^2)y \, dy$$

$$= \left[\frac{1}{4}y^2 - \frac{1}{8}y^4\right]_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$