MATH 318 — Final exam solutions — 2019

Note: These solutions may not include all details expected from you.

1. A random variable X has p.d.f. $\frac{x+2}{4}$ on $[-1,1]$ and 0 outside $[-1,1]$.
   (a) What is the expectation $\mathbf{E}X$?
   (b) What is the variance $\operatorname{Var} X$?
   (c) What is the characteristic function of X?

Solution: Compute the following integrals:
(a) $E[X] = \int_{-1}^{1} \frac{x+2}{4}\, x\, dx$.
(b) $\operatorname{Var}(X) = \int_{-1}^{1} \frac{x+2}{4}\, x^2\, dx - (E[X])^2$.
(c) $\phi(t) = \int_{-1}^{1} \frac{x+2}{4}\, e^{itx} dx$.

2. A committee of 7 people is chosen randomly out of a group of 60. Of the 60, Dorothy has 20 friends and 15 enemies. Let X be the number of friends of Dorothy in the committee and Y the number of enemies.
   (a) Find $P(X \geq 5)$, and $P(Y \geq 5)$.
   (b) Find $P(\{X \geq 5\} \cup \{Y \geq 5\})$?
   (c) What is $E(X - Y)$?
   (d) What is $E(X|Y = y)$?

Solution:
(a) $P(X \geq 5) = \frac{\binom{20}{5}\binom{40}{2}+\binom{20}{6}\binom{40}{1}+\binom{20}{7}\binom{40}{0}}{\binom{60}{7}}$. For Y, replace $20, 40$ by $15, 45$.
(b) By inclusion-exclusion, this is $P(X \geq 5) + P(Y \geq 5) - P(X, Y \geq 5)$. The last is 0, so tadd up the two answers for (a).
(c) Each has probability $1/3$ of being a friend and $1/4$ of being an enemy, so $E[X] = 7/3$ and $E[Y] = 7/4$, and the difference is $E[X - Y] = 7/12$.
(d) If y of the seven are enemies, each remaining one has probability $20/45$ of being a friend, so $E(X|Y = y) = \frac{20(7-y)}{45}$.

3. Let X, Y be independent exponential variables with parameters λ. Compute all integrals in the following.
   (a) Find $E[XY]$.
   (b) Find $P(X > Y + 2)$.

(c) Find the density function of $X - Y$.
(d) Find the characteristic function of $X - Y$.

Solution:

(a) $E[XY] = E[X]E[Y] = 1/\lambda^2$.
(b) $P(X \geq Y + 2) = \int_0^\infty \int_y +2^\infty \lambda^2 e^{-\lambda x}\, dx\, dy = e^{-2\lambda}/2$.
(c) This is $\phi_X(t)\phi_Y(-t) = \frac{\lambda}{\lambda - it}\frac{\lambda}{\lambda + it} = \frac{\lambda^2}{\lambda^2 + t^2}$.

4. Customers arrive at a store according to a Poisson process with rate $\lambda = 6$ per hour. The store is open from 8:00 to 18:00.

   (a) What is the probability that at least two customers arrive between 8:00 and 8:30?
   (b) Use the Central limit theorem to estimate the probability that at most 50 customers come in a day. You may use Φ in your answer.
   (c) What is the expected number of customers arriving before noon?
   (d) What is the expected number of customers arriving all day, conditioned on 30 customers arriving before noon?

Solution:

(a) $P(\text{Poi}(3) \geq 2) = 1 - e^{-3} - 3e^{-3}$.
(b) In a day we have $X = \text{Poi}(60) \approx N(60, 60)$. Therefore $P(X \leq 50) \approx \Phi(-10/60)$. (A better approximation is $\Phi(-9.5/60)$.
(c) 24.
(d) 30 before noon, plus an independent Poi(36) after noon gives 66.

5. Jack and Jill are magicians, who perform the following trick. A volunteer picks a card, and Jack or Jill makes a prediction what the card is.

   (a) Jack is not very talented, and the trick succeeds only 1% of the time. Use the Poisson approximation to estimate the probability that the trick succeeds exactly three times out of 200 attempts.
   (b) Jill succeeds at the same trick 200 out of 1000 independent attempts, give a 95% confidence interval for the probability that she succeeds at each attempt.

Solution:

(a) This is $P(\text{Poi}(2) = 3) = e^{-2}2^3/3!$.
(b) The measurement is 0.2. If the actual value is p then its Variance is $\sqrt{p(1-p)/n}$. Since p is roughly 0.2, this is $\sqrt{(0.2)(0.8)/1000)} = \sqrt{0.00016}$. The confidence interval is $0.2 \pm (1.96)\sqrt{0.00016}$.

6. Biking to work takes Martha a normal $N(23, 2^2)$ number of minutes. The subway takes a fixed 15 minutes, plus the random waiting time for the train which is Exp(0.2). Martha takes the bus with probability 1/3 each day. If the trip takes more than 25 minutes, she is late.

   (a) What is the probability that Martha is late for work on any day?
   (b) What is the probability she took the bus, conditioned on her being late?
   (c) What is the Variance of Martha's travel time?

Solution:

(a) When cycling it is $1 - \Phi(1)$. With the bus it is e^{-2}. Together it is $(2/3)(\Phi(-1)) + (1/3)e^{-2}$.
(b) By Bayes: $\frac{(1/3)e^{12}}{(2/3)(\Phi(-1))+(1/3)e^{-2}}$
(c) The expectation is 22. Also, $\mathbf{E}X^2 = (2/3)(23^2 + 2^2) + (1/3)(15^2 + 15 \cdot 5 + 2 \cdot 5^2)$, so the variance i this minus 22^2.

7. A random variable X has characteristic function $\phi(t) = \frac{\cos(t)}{1+t^2}$. Compute the following: (a) $E[X]$. (b) $\text{Var}(X)$. (c) $E[X^4]$.

Solution: The Taylor expansion is $\phi(t) = 1 - (3/2)t^2 + (3/4)t^4 + O(t^6)$. This gives $E[X] = 0$, $E[X^2]/2 = 3/2$ and $E[X^4]/4! = 3/4$. Therefore $\text{Var}(X) = 3$ and $E[X^4] = 18$.

8. Consider a Markov chain with states $\{1, 2, 3, 4, 5, 6\}$ and the following transition probability matrix:

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

(a) Draw a transition diagram for the Markov chain.
(b) If $X_0 = 5$, what is the distribution of X_2 (write it as a vector)?
(c) What are the communicating classes? For each communicating class, determine if it is transient or recurrent and whether it is periodic or aperiodic.
(d) Find a stationary distribution with $\pi_6 = 0$ (write it as a vector).
(e) If $X_0 = 1$, what is the expected time before the chain returns to 1?

Solution:

(a) Drawing.
(b) The 5th row of P^2 is $(1/6, 0, 1/6 + 1/9, 1/6, 1/9, 1/9 + 1/6)$.
(c) $\{1, 2, 3\}$ is recurrent aperiodic. $\{4, 6\}$ as well. $\{5\}$ is transient, aperiodic.
(d) Since $\pi_6 = 0$, also $\pi_4 = \pi_5 = 0$. Solve $\pi = \pi$ with $\pi_1 + \pi_2 + \pi_3 = 1$ gives $\pi = (2/7, 3/7, 2/7, 0, 0, 0)$.
(e) This is $1/\pi_1 = 7/2$.

9. A random walk on a graph has state space the nodes of the graph. From each node, it moves to a connected node, with equal probability for each connected node. (If x has degree d_x, and x, y are connected, then $P_{x,y} = 1/d_x$.)

(a) Define: a Markov chain is reversible with respect to a measure π.
(b) Show that the random walk on the graph below is reversible with respect to some distribution π, and find that π.
(c) Find the asymptotic fraction of time the random walk spends at the topmost vertex in this graph.

Solution:

(a) $\pi_x P_{xy} = \pi_y P_{yx}$ for every pair of states.
(b) If x, y are connected, detailed balance gives $\pi_x/d_x = \pi_y/d_y$, so $\pi_x = cd_x$ for some c, which must be the sum of the degrees. This choice of π works.
(c) This is $\pi_x = 2/20$ in this case.