Math 318 – homework 7 solutions

Problem 1. (a) If X is an integer valued random variable, show that the characteristic function of X has period 2π.

(b) Prove the converse: if $\phi(t) = \phi(t + 2\pi)$ for every t, then show that X takes only integer values. (Hint: If a random variable Y satisfies $Y \geq 0$ and $E[Y] = 0$ then $P(Y = 0) = 1$. Use this for a carefully chosen function of X. Hint: consider $\phi(2\pi)$.)

Solution.

(a) Since X is an integer, $e^{2\pi i X} = 1$. Therefore, $e^{itX} = e^{i(t+2\pi)X}$, and so they have the same expectation. Note that there can be other periods, e.g. if X is always even there is period π.

(b) Periodicity implies $\phi(2\pi) = \phi(0) = 1$, so $Ee^{2\pi i X} = 1$. To deduce that X must be an integer, we use the hint. Considering the real part, we have $E\cos(2\pi X) = 1$, but $\cos(2\pi X) \leq 1$. The only way for a random variable that is always at most 1 to have expectation 1 is if it is always equal to 1. Therefore we always have $\cos(2\pi X) = 1$ and so X is an integer.

Problem 2. Let U, V be two independent random variables, with $E[U] = E[V] = 0$, and let $X = U + V$. Assume also that U and V have finite moments. Find an expression for $E[X^3]$ and for $E[X^4]$, in terms of moments of U, V.

Solution. Use $(U + V)^3 = U^3 + 3U^2V + 2UV^2 + V^3$, and take expectation of each term. Since U, V are independent and $E[U] = E[V] = 0$, this gives

$$E(U + V)^3 = E[U^3] + E[V^3].$$

Similarly, use $(U + V)^4 = U^4 + 4U^3V + 6U^2V^2 + 4UV^3 + V^4$ to find

$$E(U + V)^4 = E[U^y] + 6E[U^2]E[V^2] + E[V^4].$$

Problem 3. Let A be an $n \times n$ matrix, and let $X = (X_1, \ldots, X_n)^T$ be a vector of i.i.d. $N(0, 1)$ random variables. Let $Y = AX$, and Y_i its coordinates.

(a) What is the distribution of Y_i?

(b) Find $\mathrm{Cov}(Y_i, Y_j)$.

(c) If $n = 2$ and A is invertible, show that Y_1, Y_2 have joint probability density function

$$\frac{1}{(2\pi)^{n/2}|\det A|} e^{-(y^T A^{T^{-1}} A^{-1} y)/2}$$

on $\mathbb{R}^2$, where $y = (y_1, y_2)^T$ is a vector.) (Hint: What is the Jacobian of the mapping from X to Y?)

bonus For any n, if A is invertible, show that Y has probability density function with the same formula.

Solution.

(a) $Y_i = \sum_k A_{ik} X_k$ is a sum of independent normal variables, so has distribution $N(0, \sum_k A_{ik}^2)$.

(b) Using the sums for Y_i and Y_j we get

$$\begin{aligned}
\mathrm{Cov}(Y_i, Y_j) &= \sum_{k,\ell} \mathrm{Cov}(A_{ik} X_k, A_{i\ell} X_\ell) \\
&= \sum_{k,\ell} A_{ik} A_{j\ell} \mathrm{Cov}(X_k, X_\ell) \\
&= \sum_k A_{ik} A_{jk},
\end{aligned}$$

since the covariance is 0 unless $\ell = k$ and is 1 if $\ell = k$.

(c) The Jacobian of the map $Y = AX$ is $|\det A|$. We have $X = A^{-1}Y$, and so $\sum X_i^2 = X^T X = Y^T BY$ with $B = (A^{-1})^T A^{-1}$. The joint density of X is $\frac{1}{(2\pi)^{n/2}} e^{-X^T X/2}$, and so the change of variable formula gives the claimed density of Y.

Problem 4. Cosider the random walk in one dimension $\mathbb{Z}$ with bias. We will consider $p = 0.5$, $p = 0.51$, and $p = 0.502$. Let $X_i = \pm 1$ with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ be independent. Let $S_n = X_1 + X_2 + \cdots + X_n$. For each of these, do the following.

(a) Simulate and plot a random walk with 10^6 steps. Submit your code and a plot of $S_0, \ldots, S_n$.

(b) Simulate 1000 independent random walks, each for up to 10^6 steps. For each walk, let $T > 0$ be the first time the walk returns to 0. If a walk does not return to 0, let $T = 10^6$. For each of the p's, how many of the 1000 walks did not return to 0? Submit histograms of the values of T observed.

(c) Make a log-log plot of the fraction of times $T > k$ for $k = 0, \ldots, 10^6$.

(d) Based on the previous plot, guess the asymptotics of $P(T > n)$ as $n \to \infty$. What do you think is the mean of T if there was no limit of 10^6?

Problem 5. This problem investigates the recurrence of 2-dimensional random walk. Let (X_n, Y_n) be a two dimensional random walk. At each step pick either X or Y and either increase or decrease it by 1, all with equal probabilities, and independeent of other steps.

(a) Simulate this process for 10^6 steps. Submit code and a plot of the walk in the $X - Y$ plane (there is no coordinate for time, though you could use colour to indicate the time a point is visited.

(b) Simulate 1000 random walks for up to 10^6 steps, and for each keep track of the number of steps before it returns to 0 for the first time. How many of the walks failed to return to 0? Is this consistent with the theorem that the random walk is recurrent?

(c) Make a histogram of the resulting return times for the walks that did return, on a logarithmic scale. Can you guess how fast $P(T > n)$ decays from this?

Note. it is much faster to only generate each random walk until it returns to 0, and stop there. Otherwise this make take a long time to run.