PHYS 250 2021 Final Exam Solutions

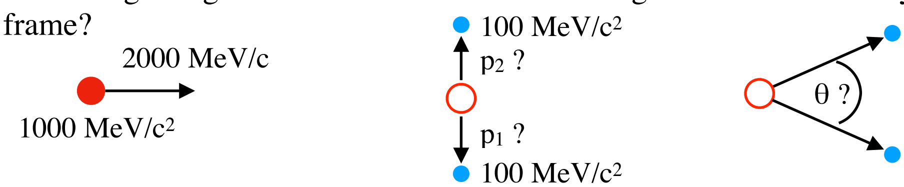
2. (4 points) A spaceship departs toward Earth from a star that is 100 light-years away at 99% of the speed of light. How long does the trip to Earth take in the frame of the spaceship?

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.99^2}} = 7.089$$

In the spaceship frame, the Earth looks $\dfrac{100}{\gamma} = \dfrac{100}{7.089} = 14.11$ light years away,

and is approaching at $\beta = 0.99$. Earth will arrive in $\dfrac{14.11}{\beta} = 14.25$ years.

3. (4 points) Particle A with rest mass of 1000 MeV/c^2 with momentum of 2000 MeV/c in the lab frame decays into two particles with mass 100 MeV/c^2. In the frame of particle A, the decay particles travel at right angles to its motion. What is the angle between the decay particle tracks in the lab frame?



Initial state in lab frame   Final state in decay frame   Final state in lab frame

In the frame of particle A, the equal-mass decay products each have total energy equal to half the mass of particle A, or $E' = 500$ MeV/c^2.

They have momentum $p' = \sqrt{E^2 - m^2} = \sqrt{500^2 - 100^2} = 489.9$ MeV/c.

The 90° decay angle means that the momentum along the original direction of A is zero, and it is entirely transverse.

In the lab, particle A had energy $E = \sqrt{p^2 + m^2} = \sqrt{2000^2 + 1000^2} = 2236$ MeV.

Therefore it had $\gamma = \dfrac{E}{m} = \dfrac{2236}{1000} = 2.236$ and $\beta\gamma = \dfrac{p}{m} = \dfrac{2000}{1000} = 2.000$.

In the lab, the decay particles have momentum along the original A direction of

$p = \gamma p' + \beta\gamma E' = \gamma \cdot 0 + 2.000 \cdot 500 = 1000$ MeV/c.

They have transverse momentum in the lab of ± 489.9 MeV/c,

so they have angles of $\tan^{-1}\dfrac{\pm 489.9}{1000} = \pm 26.10° = \pm 0.4555$ radians.

So the lab opening angle is $52.20° = 0.9111$ radians.

4. (4 points) What is the Bragg scattering angle for photons with energy of 10 keV incident on a crystal with plane spacing of 0.100 nm ?

$$\text{Planck-Einstein: } E = hf = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{E} = \frac{1240 \text{ eV-nm}}{10\times 10^3 \text{ eV}} = 0.1240 \text{ nm}$$

$$\text{Bragg: } n\lambda = 2d\sin\theta \rightarrow \sin\theta = \frac{\lambda}{2d} = \frac{0.1240 \text{ nm}}{2\cdot 0.100 \text{ nm}} = 0.6200$$

$$\rightarrow \theta = 38.32° = 0.6687 \text{ radians}$$

5. (4 points) What is the Bragg scattering angle for electrons with kinetic energy of 10 keV incident on a crystal with plane spacing of 0.100 nm ?

$$\text{Classical } E = \frac{1}{2}mv^2 = \frac{p^2}{2m} \rightarrow p = \sqrt{2mE}$$

$$\text{de Broglie: } \lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{hc}{\sqrt{2mc^2E}} = \frac{1240 \text{ eV-nm}}{\sqrt{2\cdot 0.511\times 10^6 \text{ eV}\cdot 10\times 10^3\text{eV}}} = 0.01227 \text{ nm}$$

$$\text{Bragg: } n\lambda = 2d\sin\theta \rightarrow \sin\theta = \frac{\lambda}{2d} = \frac{0.01227 \text{ nm}}{2\cdot 0.100 \text{ nm}} = 0.06133$$

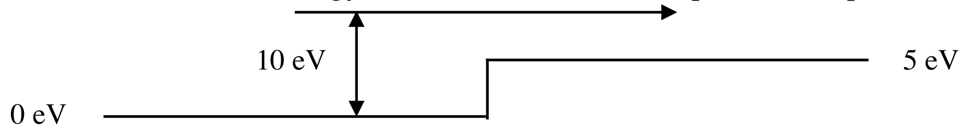
$$\rightarrow \theta = 3.516° = 0.06137 \text{ radians}$$

6. (4 points) What is the wavelength of the n = 4 to n = 2 transition in Li^{+2} ions?

$$\text{Rydberg: } \frac{1}{\lambda} = R\cdot Z^2\cdot\left(\frac{1}{n_2^2}-\frac{1}{n_1^2}\right) = \frac{1}{91.13 \text{ nm}}\cdot 3^2\cdot\left(\frac{1}{2^2}-\frac{1}{4^2}\right) = 1.852\times 10^{-2} \text{ nm}^{-1}$$

$$\lambda = 54.00 \text{ nm}$$

7. An electron with kinetic energy of 10 eV is incident on a potential step of 5 eV.



A. (4 points) What is the amplitude of the transmitted wave relative to the incident wave?

$$k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2mc^2E}}{\hbar c} = \frac{\sqrt{2 \cdot 0.511 \times 10^6 \text{ eV} \cdot 10 \text{ eV}}}{197.3 \text{ eV-nm}} = 16.20 \text{ nm}^{-1}$$

$$k' = \frac{\sqrt{2m(E-V)}}{\hbar} = \frac{\sqrt{2 \cdot 0.511 \times 10^6 \text{ eV} \cdot (10 \text{ eV} - 5 \text{ eV})}}{197.3 \text{ eV-nm}} = 11.46 \text{ nm}^{-1}$$

$$T = \frac{2k}{k+k'} = \frac{2 \cdot 16.20}{16.20 + 11.46} = 1.171$$

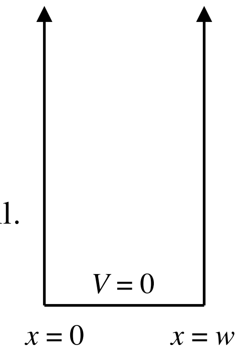
Bigger than the incident wave!

B. (4 points) What is the probability that the electron is transmitted?

The transmission probability is NOT just the square of the transmitted amplitude.
because the velocity changes on the other side of the step.
It's simplest to calculate the reflection amplitude, where there is no velocity change,
square it, and subtract from 1 to get the transmission probability.

$$R = \frac{k-k'}{k+k'} = \frac{16.20 - 11.46}{16.20 + 11.46} = 0.1714$$

Transmission probability $= 1 - 0.1714^2 = 0.9706$



8. A particle with mass *m* is in a one-dimensional infinite square well with width *w*.

A. (4 points) Write the time-independent wavefunction for the ground state inside the well.

$\psi(x) = A\sin kx$ with $k = \dfrac{\pi}{w}$ or equivalently $\psi(x) = A\sin\dfrac{\pi x}{w}$

B. (4 points) Write the time-dependent wavefunction for the ground state inside the well.

$$\psi(x,t) = \psi(x)\cdot\exp(-i\omega t) \text{ with } \hbar\omega = E = \frac{p^2}{2m} = \frac{(\hbar k)^2}{2m} \rightarrow \omega = \frac{\hbar\pi^2}{2mw^2} \text{ so}$$
$$\psi(x,t) = A\sin(kx)\cdot\exp\left(-i\frac{\hbar k^2}{2m}t\right) = A\sin\left(\frac{\pi}{w}x\right)\cdot\exp\left(-i\frac{\hbar\pi^2}{2mw^2}t\right)$$

C. (4 points) Re-write your answer for B as a superposition of time-dependent free-particle wavefunctions. Hint: $\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2}$ and $\sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$

$$\sin(kx) = \frac{\exp(ikx)-\exp(-ikx)}{2i}$$
$$\psi(x,t) = A\frac{\exp(ikx)-\exp(-ikx)}{2i}\cdot\exp\left(-i\frac{\hbar k^2}{2m}t\right)$$
$$= \frac{A}{2i}\left\{\exp\left(i\left[kx-\frac{\hbar k^2}{2m}t\right]\right)-\exp\left(i\left[-kx-\frac{\hbar k^2}{2m}t\right]\right)\right\}$$
$$= \frac{A}{2i}\exp\left(i\left[kx-\frac{\hbar k^2}{2m}t\right]\right)-\frac{A}{2i}\exp\left(i\left[-kx-\frac{\hbar k^2}{2m}t\right]\right) \text{ or equivalently}$$
$$\psi(x,t) = \frac{A}{2i}\exp\left(i\left[\frac{\pi}{w}x-\frac{\hbar\pi^2}{2mw^2}t\right]\right)-\frac{A}{2i}\exp\left(i\left[-\frac{\pi}{w}x-\frac{\hbar\pi^2}{2mw^2}t\right]\right)$$

D. (4 points) What would be the result if you measured the momentum of the particle in the ground state of the infinite square well?

The wavefunction has be re-written as a superposition of one wave with momentum $p = \hbar k = \dfrac{\hbar\pi}{w}$

and another with equal amplitude and momentum $p = \hbar(-k) = -\dfrac{\hbar\pi}{w}$.

So the only possible results are $p = \dfrac{\hbar\pi}{w}$ and $p = -\dfrac{\hbar\pi}{w}$, and they are equally likely.

9. (10 points) For time-independent $\psi(r,\theta,\phi) = F(r)\cdot G(\theta)\cdot H(\phi)$,
write the differential equations that must be solved for F, G, and H.
You don't have to solve them, or write the solutions.
But do explain the meaning of any constants you introduce.

$\dfrac{\partial^2 H(\phi)}{\partial\phi^2} = -\mu H(\phi)$ where μ is an eigenvalue we must solve for.

$\sin\theta \dfrac{\partial}{\partial\theta}\left[\sin\theta \dfrac{\partial G(\theta)}{\partial\theta}\right] = \left[\mu - \lambda\sin^2\theta\right]G(\theta)$ where λ is another eigenvalue.

$-\dfrac{\hbar^2}{2M}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left[r^2\dfrac{\partial F(r)}{\partial r}\right] + \left[V(r) + \dfrac{\hbar^2\lambda}{2Mr^2}\right]F(r) = E\cdot F(r)$ where E is another eigenvalue.

The last equation can be simplified by $F(r) = \dfrac{1}{r}\cdot U(r)$ into

$$-\frac{\hbar^2}{2M}\frac{\partial^2 U(r)}{\partial r^2} + \left[V(r) + \frac{\hbar^2\lambda}{2Mr^2}\right]U(r) = E\cdot U(r)$$

10. An electron is inside cubic box with 10 nm edges with zero potential inside and infinite potential outside.

A. (4 points) What is the ground state energy?

For a rectangular box potential,

$$E = \frac{(\pi\hbar)^2}{2m}\left(\frac{n_X^2}{w^2} + \frac{n_Y^2}{w^2} + \frac{n_Z^2}{w^2}\right) = \frac{(\pi\hbar c)^2}{2mc^2w^2}\left(n_X^2 + n_Y^2 + n_Z^2\right)$$

$$\frac{(\pi\hbar c)^2}{2mc^2w^2} = \frac{(\pi \cdot 197.3\ \text{eV-nm})^2}{2\cdot 0.511\times 10^6\ \text{eV}\cdot (10\ \text{nm})^2} = 3.759\times 10^{-3}\ \text{eV}$$

For the ground state, $n_X = n_Y = n_Z = 1$, so

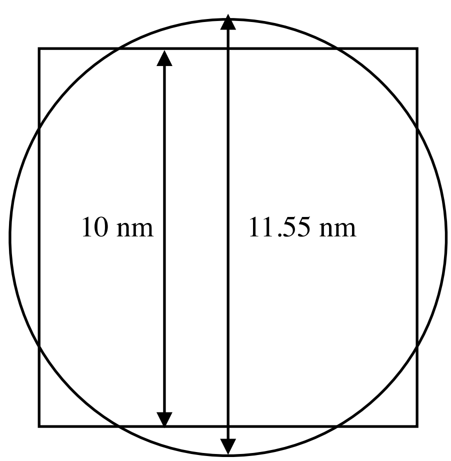
$$E = 3\cdot 3.759\times 10^{-3}\ \text{eV} = 1.128\times 10^{-2}\ \text{eV}$$

B. (4 points) What is the radius for a spherical potential well with zero potential inside and infinite potential outside with the same ground state energy?

For a spherical square well, the $\ell = 0$ energies are $E = n^2\dfrac{(\pi\hbar)^2}{2mr^2}$.

Setting ground state energies equal, $E = 1\cdot\dfrac{(\pi\hbar)^2}{2mr^2} = 3\cdot\dfrac{(\pi\hbar)^2}{2mw^2} \rightarrow \dfrac{1}{r^2} = \dfrac{3}{w^2} \rightarrow r = \dfrac{w}{\sqrt{3}}$

So the radius is $r = \dfrac{10\ \text{nm}}{\sqrt{3}} = 5.774$ nm, and diameter 11.55 nm.



11. (4 points)

What are the n, ℓ, and m quantum numbers for $\psi(r,\theta,\phi) = r^2 \exp\left(-\frac{r}{3a_0}\right)\sin^2\theta \exp(-2i\phi)$?

This looks like a hydrogen wavefunction.
The $3a_0$ in the exponential implies $n = 3$.
The r^2 factor implies $U(r) = r^3$ which is consistent.
The $\sin^2\theta$ factor implies $\ell = 2$.
The $-2i\phi$ exponent implies $m = -2$.

14. A particle with mass m is in a <u>one</u>-dimensional harmonic oscillator with spring constant k.

A. (4 points) What is the energy of the first excited state?

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \text{ with } \omega = \sqrt{\frac{k}{m}} \quad \text{so } E_{n=1} = \frac{3}{2}\hbar\omega$$

B. (4 points) Write the wavefunction describing the first excited state.
Don't worry about normalization.

$$\psi_{n=1}(x) = x\exp\left(-\frac{x^2}{2b^2}\right) \text{ with } b^2 = \frac{\hbar}{\sqrt{km}} \quad \text{or} \quad \psi(x) = x\exp\left(-\frac{x^2\sqrt{km}}{2\hbar}\right)$$