PHYS 250 2023 Final Exam Solutions

1. (6 points) A GPS satellite orbits with a velocity of 3.9 km/s. How many seconds per day does its clock lose relative to the ground? (Zero is not acceptable). How far does light travel in that time?

$$\beta = \frac{3.9\times10^3 \text{ m/s}}{2.997\times10^8 \text{ m/s}} = 1.301\times10^{-5} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1+\frac{1}{2}\beta^2 \quad \Delta t = 24\cdot60\cdot60\cdot\left(\frac{1}{\gamma}-1\right) = 24\cdot60\cdot60\cdot\left(\frac{1-\gamma}{\gamma}\right)$$

$$1-\gamma \approx -\frac{1}{2}\beta^2 = -8.463\times10^{-11} \qquad \Delta t = -7.312\times10^{-6} \text{ sec/day}$$

$$L = \Delta t\cdot c = 7.312\times10^{-6}\cdot 2.998\times10^8 = 2192 \text{ meters}$$

2. (6 points) The PEP-II accelerator collided 9 GeV electrons with 3 GeV positrons head on. What was the total energy in the centre of mass?

For GeV electrons, $E = p$ $\quad \underset{\rightarrow}{P}_{e^-} = (9,9) \quad \underset{\rightarrow}{P}_{e^+} = (3,-3)$

$$E_{cm}^2 = \left(\underset{\rightarrow}{P}^*_{e^-} + \underset{\rightarrow}{P}^*_{e^+}\right)^2 = \left(\underset{\rightarrow}{P}_{e^-} + \underset{\rightarrow}{P}_{e^+}\right)^2 = (9+3, 9-3)^2 = 12^2 - 6^2 = 108 \rightarrow E_{cm} = 10.39 \text{ GeV}$$

3. Light with wavelength 380 nm from a Hydrogen discharge tube shines on a Cesium photocathode (work function 2.14 eV).

A. (6 points) What is the maximum photoelectron energy?

$$E_\gamma = \frac{hc}{\lambda} = \frac{1240 \text{ eV-nm}}{380 \text{ nm}} = 3.263 \text{ eV} \quad E_{e-\max} = E_\gamma - \phi_{\text{work}} = 3.263 - 2.14 = 1.123 \text{ eV}$$

B. (6 points) What are the two quantum states involved in the Hydrogen transition?

$$E_\gamma = 13.6 \text{ eV}\cdot\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) = 3.263 \text{ eV} \rightarrow \frac{1}{n_1^2}-\frac{1}{n_2^2} = \frac{3.263}{13.6} = 0.2399$$

$n_1 = 1$ gives energy much too large. $n_1 = 3$ gives energy much too small.

$$n_1 = 2 \rightarrow \frac{1}{n_2^2} = \frac{1}{2^2} - 0.2399 = 1.01\times10^{-2} \rightarrow n_2 = \sqrt{\frac{1}{1.01\times10^{-2}}} = 9.95 \rightarrow n_2 = 10$$

4. (6 points) The Kα X-ray line from the anode of an X-ray tube is Bragg-diffracted by 9.378° by a crystal with layer spacing of 70 pm. What is the atomic number Z of the anode?

$$n\lambda = 2d\sin\theta = 2\cdot70\times10^{-3} \text{ nm}\cdot\sin 9.378° = 2.281\times10^{-2} \text{ nm} = 22.81 \text{ pm}$$

$$E_\gamma = \frac{hc}{\lambda} = \frac{1240 \text{ eV-nm}}{2.281\times10^{-2} \text{ nm}} = 54.36 \text{ keV} \quad E_{K\alpha} = 13.6 \text{ eV}\cdot\left(\frac{1}{1^2}-\frac{1}{2^2}\right)\cdot(Z-1)^2 = 10.2 \text{ eV}\cdot(Z-1)^2$$

$$(Z-1)^2 = \frac{54.36 \text{ keV}}{10.2 \text{ eV}} = 5.330\times10^3 \rightarrow Z = \sqrt{5.330\times10^3}+1 = 74 \text{ (Tungsten)}$$

5. A particle with mass m has energy E in region I with $V = 0$. It enters region II at x = 0 with $V > E$. After distance L, it enters region III with $V = 0$.

A. (5 points) Write the wavefunctions for regions I, II, and III. Define any variables that you introduce, and give values where possible. You don't have to solve for all the variables yet.

$$\psi_{\text{I}} = \psi_{\text{Incident}} + \psi_{\text{Reflected}} \qquad \psi_{\text{II}} = \psi_{\text{Decaying}} + \psi_{\text{Growing}} \qquad \psi_{\text{III}} = \psi_{\text{Transmitted}}$$

$$\psi_{\text{Incident}} = e^{ikx} \quad \psi_{\text{Reflected}} = Re^{-ikx} \quad \psi_{\text{Decaying}} = De^{-k'x} \quad \psi_{\text{Growing}} = Ge^{+k'x} \quad \psi_{\text{Transmitted}} = Te^{ik\cdot(x-L)}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad k' = \frac{\sqrt{2m\cdot(V-E)}}{\hbar}$$ R, D, G, T are amplitude coefficients

B. (5 points) Write boundary condition equations whose solution determines your variables. You don't have to solve the equations.

The boundary conditions are continuity of ψ and $\frac{\partial\psi}{\partial x}$ at the boundaries.

ψ at $x = 0$: $1 + R = D + G$

$\frac{\partial\psi}{\partial x}$ at $x = 0$: $ik - ikR = -k'D + k'G$

ψ at $x = L$: $De^{-k'L} + Ge^{+k'L} = T$

$\frac{\partial\psi}{\partial x}$ at $x = L$: $-k'De^{-k'L} + k'Ge^{+k'L} = -ikT$

$$\left[\begin{array}{cccc:c} R & D & G & T & \\ \hline 1 & -1 & -1 & 0 & -1 \\ -ik & k' & -k' & 0 & -ik \\ 0 & e^{-k'L} & e^{+k'L} & -1 & 0 \\ 0 & -k'e^{-k'L} & k'e^{+k'L} & -ik & 0 \end{array}\right]$$

C. (5 points) If the particle is an electron with kinetic energy E = 1 eV, and the potential in region II is V = 5 eV and length L is 1 nm, what is the approximate probability that the particle reaches region III?

The tunnelling probability for a flat potential with sharp steps is

$$P = G\cdot\exp(-2\kappa L) \text{ with } \kappa = \frac{\sqrt{2m(V-E)}}{\hbar} \text{ and } G = 16\frac{E}{V}\cdot\left(1-\frac{E}{V}\right)$$

$$\kappa = \frac{\sqrt{2mc^2}}{\hbar c}\sqrt{V-E} = \frac{\sqrt{2\cdot 0.511\times10^6 \text{ eV}}}{197.4 \text{ eV-nm}} = \frac{5.121}{\sqrt{\text{eV}}\text{-nm}}\cdot\sqrt{5 \text{ eV} - 1 \text{ eV}} = 10.24 \text{ nm}^{-1}$$

$$\exp(-2\kappa L) = \exp\left(-2\cdot 10.24 \text{ nm}^{-1}\cdot 1 \text{ nm}\right) = 1.269\times10^{-9}$$

$$G = 16\cdot\frac{1 \text{ eV}}{5 \text{ eV}}\cdot\left(1-\frac{1 \text{ eV}}{5 \text{ eV}}\right) = 2.560 \quad P = 3.248\times10^{-9}$$ Exact from above is $|T|^2 = 3.252\times10^{-9}$

6. Harmonic Oscillator

A. (5 points) Write the time-independent Schrodinger Equation for a harmonic oscillator with spring constant k and mass m in 1 dimension.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}kx^2\psi = E\psi$$

B. (5 points) Write the wavefunctions for the two lowest energy states. Define any variables you introduce. Don't worry about normalization.

$$\psi_0(x)=\exp\left(-\frac{y^2}{2}\right)\quad \psi_1(x)=y\cdot\exp\left(-\frac{y^2}{2}\right)\text{ with } y=\frac{x}{b},\ b^2=\frac{\hbar}{\sqrt{km}}$$

$$\psi_0(x)=\exp\left(-\frac{x^2\sqrt{km}}{2\hbar}\right)\quad \psi_1(x)=x\cdot\exp\left(-\frac{x^2\sqrt{km}}{2\hbar}\right)$$

C. (5 points) If the particle is an electron, and the spring constant of the potential is 16 eV/nm^2, what are the energies in eV of the 3 lowest states?

$$E_n=\left(n+\frac{1}{2}\right)\cdot\hbar\cdot\sqrt{\frac{k}{m}}\qquad \hbar\cdot\sqrt{\frac{k}{m}}=\frac{\hbar c}{\sqrt{mc^2}}\cdot\sqrt{k}=\frac{197.4\text{ eV-nm}}{\sqrt{0.511\times10^6\text{ eV}}}\cdot\sqrt{16\text{ eV/nm}^2}=1.104\text{ eV}$$

$$E_0=0.5522\text{ eV}\quad E_1=1.657\text{ eV}\quad E_2=2.761\text{ eV}$$

7. A particle is in a rectangular box with $V = 0$ for $0 < x < w_x$ and $0 < y < w_y$ and $0 < z < w_z$ and $V = \infty$ elsewhere.

A. (5 points) What are the boundary conditions on the wavefunction?

$\psi = 0$ at $x = 0,\ x = w_x,\ y = 0,\ y = w_y,\ z = 0,\ z = w_z$

B. (5 points) Write the time-independent wavefunction for the particle. Define any variables you introduce. Don't worry about normalization.

$$\psi_{nx,ny,nz} = \sin\left(n_x \frac{\pi x}{w_x}\right)\cdot\sin\left(n_y \frac{\pi y}{w_y}\right)\cdot\sin\left(n_z \frac{\pi z}{w_z}\right) \quad n_x, n_y, n_z = 1,2,3,\ldots \text{ in any combination}$$

C. (5 points) Write the formula for the energies of the possible states.

$$E_{nx,ny,nz} = \frac{\hbar^2\pi^2}{2m}\cdot\left[\frac{n_x^2}{w_x^2} + \frac{n_y^2}{w_y^2} + \frac{n_z^2}{w_z^2}\right]$$

D. (5 points) If the particle is an electron, and the dimensions are $w_x = w_y = 1$ nm and $w_z = 2$ nm, what are the 6 lowest <u>energies</u> in eV ? (Not necessarily the energies of the 6 lowest <u>states</u>, since some states may have the same energy). ?

$$E_{nx,ny,nz} = \frac{(\hbar c)^2\pi^2}{2mc^2}\cdot\left[\frac{n_x^2}{w_x^2} + \frac{n_y^2}{w_y^2} + \frac{n_z^2}{w_z^2}\right] = \frac{(197.4\text{ eV-nm})^2\pi^2}{2\cdot 0.511\times 10^6\text{ eV}}\cdot\left[\frac{n_x^2}{1\text{ nm}^2} + \frac{n_y^2}{1\text{ nm}^2} + \frac{n_z^2}{2^2\text{ nm}^2}\right]$$

$$= \left[n_x^2 + n_y^2 + \frac{n_z^2}{4}\right]\cdot 0.3763\text{ eV} \quad \text{Incrementing } n_z \text{ gives the smallest energy steps}$$

$$111:\left[1+1+\frac{1^2}{4}\right] = 2.25 \quad 112:\left[1+1+\frac{2^2}{4}\right] = 3.0 \quad 113:\left[1+1+\frac{3^2}{4}\right] = 4.25$$

$$114:\left[1+1+\frac{4^2}{4}\right] = 6.0 \quad 115:\left[1+1+\frac{5^2}{4}\right] = 8.25 \quad 221:\left[4+4+\frac{1^2}{4}\right] = 8.25$$

$$211:\left[4+1+\frac{1^2}{4}\right] = 5.25 \quad 212:\left[4+1+\frac{2^2}{4}\right] = 6.0 \quad 213:\left[4+1+\frac{3^2}{4}\right] = 7.25 \quad 214:\left[4+1+\frac{4^2}{4}\right] = 9.0$$

$$E_{111} = 2.25\cdot 0.3763 = 0.847\text{ eV} \quad E_{112} = 3.00\cdot 0.3763 = 1.129\text{ eV} \quad E_{113} = 4.25\cdot 0.3763 = 1.599\text{ eV}$$

$$E_{211} = E_{121} = 5.25\cdot 0.3763 = 1.976\text{ eV} \quad E_{114} = E_{212} = E_{122} = 6.00\cdot 0.3763 = 2.258\text{ eV}$$

$$E_{213} = E_{123} = 7.25\cdot 0.3763 = 2.728\text{ eV} \quad E_{115} = E_{221} = 8.25\cdot 0.3763 = 3.102\text{ eV}$$

8. Hydrogen Atom

A. (5 points). Write the energy of the ground state of Hydrogen, and the Bohr radius, in terms of fundamental constants, and in electron Volts.

$$E_{100} = -\frac{m_e}{2}\cdot\left(\frac{q_e^2}{4\pi\varepsilon_0\hbar}\right)^2 = -13.6 \text{ eV} \quad a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e q_e^2} = \frac{q_e^2}{8\pi\varepsilon_0 E_{100}} = 52.97 \text{ pm}$$

B. (5 points) Write the reduced radial Schrodinger Equation that must be solved to find the Hydrogen energy levels.

$$-\frac{\hbar^2}{2M}\frac{\partial^2 U(r)}{\partial r^2} + \left(\frac{\hbar^2\cdot\ell\cdot(\ell+1)}{2Mr^2} - \frac{q^2}{4\pi\varepsilon_0 r}\right)U(r) = EU(r)$$

C. (5 points) Write the general form for the Hydrogen wavefunction, in terms of the solutions to the reduced radial Schrodinger Equation, and other functions, including all relevant quantum indices. You don't need to give details about the form of either the reduced radial solutions, or the other functions. Don't worry about normalization.

$\psi_{n\ell m}(r,\theta,\phi) = \frac{U_{n\ell}(r)}{r}\cdot Y_\ell^m(\theta,\phi)$ where $U_{n\ell}(r)$ is a solution to the above,

and $Y_\ell^m(\theta,\phi)$ is a Spherical Harmonic. $e^{im\phi}\cdot P_\ell^m(\theta)$ Legendre function is OK

n is the "principal" index, ℓ and m are angular indices.

$\psi_{n\ell m}(r,\theta,\phi) = \exp\left(\frac{-r}{na_0}\right)\cdot\left(\frac{2r}{na_0}\right)^\ell \cdot L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na_0}\right)\cdot Y_\ell^m(\theta,\phi)$ with $L(z)$ a Laguerre polynomial, or

$\psi_{n\ell m}(r,\theta,\phi) = \exp\left(\frac{-\rho}{2}\right)\cdot\rho^\ell\cdot L_{n-\ell-1}^{2\ell+1}(\rho)\cdot Y_\ell^m(\theta,\phi)$ with $\rho = \frac{2r}{na_0}$

D. (5 points) What are n, ℓ, and m for $\psi_{n\ell m}(r,\theta,\phi) = \frac{r^2}{a_0^2}\exp\left(-\frac{r}{3a_0}\right)\sin\theta\cos\theta\, e^{-i\phi}$?

How do you know?

$n = 3$ from the denominator inside the radial exponential,

and one more than the highest power of r in the initial factor

$\ell = 2$ from the sum of the power of $\sin\theta$ and highest power of $\cos\theta$

$m = -1$ from the exponential in ϕ