1. A klystron microwave tube has a cathode at a potential of -10^6 Volts that emits a beam of electrons toward its collector (anode) at 0 Volts.

A. (1 point) What is the kinetic energy (include units) of an electron at the anode?
$10^6\text{ eV } = 1\text{ MeV } = 1.602\times10^{-13}\text{ Joules}$

B. (1 point) What is the relativistic total energy (include units) of an electron at the anode?
Add the electron rest mass contribution $mc^2 = 0.511$ MeV to get
$1.511\text{ MeV} = 1.511\times10^6\text{ eV} = 2.4206\times10^{-13}\text{ Joules}$

C. (2 points) What is the momentum (include units) of an electron at the anode?

$$pc = \sqrt{E^2 - \left(mc^2\right)^2} = \sqrt{\left(1.511\text{ MeV}\right)^2 - \left(0.511\text{ MeV}\right)^2} = 1.422\text{ MeV} = 2.278\times10^{-13}\text{ Joules}$$

$$p = 1.422\text{ MeV/c} = 7.599\times10^{-22}\text{ kg-m/s}$$

$$p = \sqrt{2mE} = \sqrt{2\cdot0.511\text{ MeV/c}^2\cdot1\text{ MeV}} = 1.011\text{ or }1.243\text{ MeV/c is wrong}$$

D. (2 points) What is the velocity (include units) of an electron at the anode?

$$E = \gamma mc^2 \quad p = \beta\gamma mc \quad \rightarrow \beta = \frac{pc}{E} = \frac{1.422\text{ MeV}}{1.511\text{ MeV}} = 0.9411$$

$$v = \beta c = 0.9411\cdot2.998\times10^8\text{ m/s} = 2.821\times10^8\text{ m/s}$$

$$v = \sqrt{2E/m} = \sqrt{2\cdot10^6\cdot1.6\times10^{-19}/9.11\times10^{-31}} = 5.93\times10^8\text{ or }7.28\times10^8\text{ m/s is wrong.}$$

$$v = \sqrt{2\cdot1\text{ MeV}/0.511\text{ MeV/c}^2} = 1.98\text{c or }2.43\text{c is wrong}$$

E. (2 points) What is the de Broglie wavelength (include units) of an electron at the anode?
de Broglie's relation $\lambda = h/p$ uses the relativistic momentum.

So $\lambda = \dfrac{hc}{pc} = \dfrac{1240\text{ eV-nm}}{1.422\times10^6\text{ eV}} = 8.720\times10^{-4}\text{ nm} = 0.8720\text{ pm}$.

It is incorrect to use the photon result $\lambda = hc/E$ with either kinetic energy giving 1.24 pm or relativistic energy giving 0.821 pm. It is incorrect to use the classical relation

$$E = \frac{p^2}{2m} \rightarrow p = \sqrt{2mE} \rightarrow \lambda = \frac{hc}{\sqrt{2mc^2E}} = \frac{1240\text{ eV-nm}}{\sqrt{2\cdot0.511\times10^6\text{ eV}\cdot E_{\text{eV}}}} = \frac{1.227\sqrt{\text{eV}}\text{-nm}}{\sqrt{E_{\text{eV}}}}$$

with either the kinetic energy (giving 1.227 pm) or the relativistic energy (giving 0.998 pm).

F. (2 points) What is the minimum wavelength (include units) of X-rays produced at the anode?

$E = hf = \dfrac{hc}{\lambda} \rightarrow \lambda = \dfrac{hc}{E}$. The minimum wavelength comes from the maximum photon energy,

which is the electron <u>kinetic</u> energy of 1 MeV, so $\lambda = \dfrac{1240\text{ eV-nm}}{10^6\text{ eV}} = 1.240\text{ pm}$.

(The de Broglie wavelength of a <u>relativistic</u> electron isn't very different than the wavelength of a photon of comparable energy).

2. A gas discharge tube produces light from singly-ionized He^{+1} that passes through a spectrometer with slits that are 1200 nm apart. It is observed that a spectral line that is diffracted by 13.176° produces photoelectrons from a metal sample, but a spectral line diffracted by 15.493° does not produce photoelectrons from the sample.

A. (2 points) What are the wavelengths (include units) of the spectral lines? Assume $n = 1$.

$$d\sin\theta = n\lambda \rightarrow \lambda = 1200\text{ nm}\cdot\sin 13.176° = 273.5\text{ nm and }1200\text{ nm}\cdot\sin 15.493° = 320.5\text{ nm}$$

$$2d\sin\theta = \lambda \rightarrow \lambda = 2400\text{ nm}\cdot\sin 13.176° = 547.1\text{ nm and }2400\text{ nm}\cdot\sin 15.493° = 641.1\text{ nm are wrong}$$

B. (2 points) What are the photon energies (include units) of the spectral lines?

$$E = \frac{hc}{\lambda} = \frac{1240\text{ eV-nm}}{273.5\text{ nm}} = 4.534\text{ eV and }E = \frac{1240\text{ eV-nm}}{320.5\text{ nm}} = 3.869\text{ eV}$$

7.263×10^{-19} and 6.198×10^{-19} Joules

C. (2 points) What can you say about the work function of the metal sample?

The work function is more than 3.869 eV and less than 4.534 eV.

D. (2 points) What transitions of singly-ionized He^{+1} are involved in the spectral lines?

The Rydberg formula for photon energy is $E = 13.60\text{ eV}\cdot Z^2\cdot\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$.

For He^{+1} with $Z = 2$, this is $E = 54.40\text{ eV}\cdot\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$.

If $n_1 = 1$ or 2, the energies would be much higher than the above.
If $n_1 = 4$, the energies would be lower than the lowest energy..

$n_1 = 3$ and $n_2 = 4$ gives $E = 54.40\cdot\left(\frac{1}{3^2}-\frac{1}{4^2}\right) = 2.644\text{ eV}$, a bit too low.

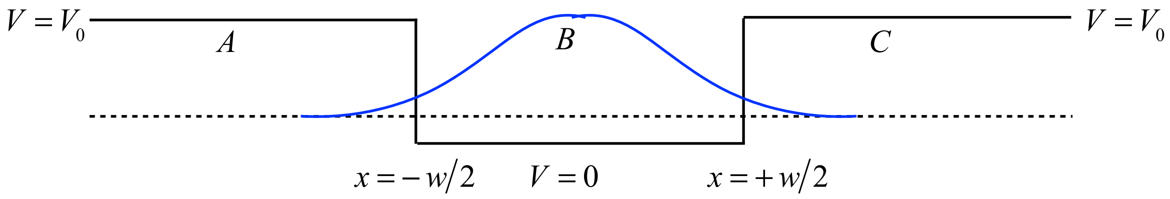
$n_1 = 3$ and $n_2 = 5$ gives $E = 54.40\cdot\left(\frac{1}{3^2}-\frac{1}{5^2}\right) = 3.868\text{ eV}$, consistent with the lower energy.

$n_1 = 3$ and $n_2 = 6$ gives $E = 54.40\cdot\left(\frac{1}{3^2}-\frac{1}{6^2}\right) = 4.533\text{ eV}$, consistent with the higher energy.

3. A potential in one dimension is $V = 0$ for $-\frac{w}{2} < x < +\frac{w}{2}$ and $V = V_0$ elsewhere.

A. (3 points) Sketch the wavefunction $\psi_1(x)$ for the lowest energy bound state
on the above diagram. The wavefunction is positive above the dotted line, negative below it.

The wavefunction is a sinusoid inside the well. The lowest energy state has roughly a half-cycle of cosine inside the well. But because the potential step is not infinite, the wavefunction does not go to zero at the steps, it extends outside as a decaying exponential.



B. (8 points) Write the wavefunctions $\psi_A(x), \psi_B(x)$, and $\psi_C(x)$
for a particle with mass m and energy $E < V_0$ in regions A, B, and C.

The functions must satisfy $\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} + V(x)\cdot\psi = E\cdot\psi$.

Inside the well, $\psi_B = \cos kx$ or $\psi_B = \sin kx$ with $k = \frac{\sqrt{2mE}}{\hbar}$

Outside the well, $\psi_A = A\exp(+k'x)$ and $\psi_C = C\exp(-k'x)$ with $k' = \frac{\sqrt{2m\cdot(V_0 - E)}}{\hbar}$

C. (8 points) Write the boundary condition equations at $x = -\frac{w}{2}$ and $x = +\frac{w}{2}$
required to solve for the bound state energy E and any other unknowns.
You don't need to solve the equations.

Continuity of the wavefunction gives

$$x = -\frac{w}{2}: \cos\left(\frac{-kw}{2}\right) = A\exp\left(\frac{+k'w}{2}\right) \quad \text{or} \quad \sin\left(\frac{-kw}{2}\right) = A\exp\left(\frac{+k'w}{2}\right) \quad \text{and}$$

$$x = +\frac{w}{2}: \cos\left(\frac{+kw}{2}\right) = C\exp\left(\frac{-k'w}{2}\right) \quad \text{or} \quad \sin\left(\frac{+kw}{2}\right) = C\exp\left(\frac{-k'w}{2}\right).$$

The derivatives are

$$\frac{d}{dx}\psi_B = -k\sin kx \text{ or } k\cos kx$$

$$\frac{d}{dx}\psi_A = Ak'\exp(+k'x) \qquad \frac{d}{dx}\psi_C = -Ck'\exp(-k'x)$$

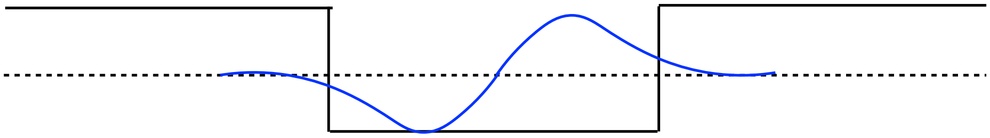
Continuity of the derivatives gives

$$x=-\frac{w}{2}:\ -k\sin\left(\frac{-kw}{2}\right) = Ak'\exp\left(\frac{+k'w}{2}\right) \text{ or } k\cos\left(\frac{-kw}{2}\right) = Ak'\exp\left(\frac{+k'w}{2}\right) \text{ and}$$

$$x=+\frac{w}{2}:-k\sin\left(\frac{+kw}{2}\right) = -Ck'\exp\left(\frac{-k'w}{2}\right) \text{ or } k\cos\left(\frac{+kw}{2}\right) = -Ck'\exp\left(\frac{-k'w}{2}\right).$$

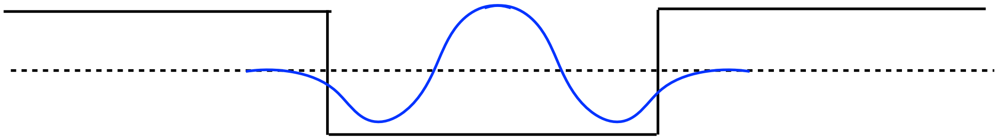
D. (3 points) Sketch the wavefunction for the first excited state $\psi_2(x)$ (assume that it is bound). The wavefunction is positive above the dotted line, negative below it.

The wavefunction will be odd, with about a full cycle of sine inside the well, and decaying exponentials outside the well, with opposite signs.



E. (3 points) Sketch the wavefunction for the next excited state $\psi_3(x)$ (assume that it is bound). The wavefunction is positive above the dotted line, negative below it.

The wavefunction will be even, with 1.5 cycles of cosine inside the well, and decaying exponentials outside the well



4. Conductivity

A. (2 points) Lithium has Z = 3. What is the quantum mechanical explanation for why it is a conductor?

For N atoms, each energy level splits into N levels, each of which can hold 2N electrons. Lithium has one electron in its 2S level, so there are N electrons in the split 2S level, which could hold 2N electrons. So there are many un-occupied split 2S levels, allowing electrons to tunnel between atoms easily.

B. (2 points) Carbon has Z = 6. What is the quantum mechanical explanation for why diamond is an insulator?

Carbon has a full “valence band” so those electrons are not free to move, and the energy gap to its “conduction band” is large compared to $k_B T$, so while electrons would be free to move there, very few electrons are found there

C. (2 points) Beryllium has Z = 4. What is the quantum mechanical explanation for why it is a conductor?

Beryllium has 2 electrons in its 2S level. When the 2S level splits into N levels, it can hold 2N electrons, and there are 2N electrons in it. That sounds like the recipe for an insulator, but it turns out that when the un-occupied 2P levels split, some of them go down in energy far enough to overlap with the split 2S levels. The overlapping energy levels allow electrons to move freely.

D. (2 points) Silicon has Z = 14 and forms the same crystal structure as diamond. What is the quantum explanation for why Silicon is a semiconductor (and diamond isn’t)?

Silicon is a semiconductor because while it has a full valence band, making it a poor conductor, the energy gap to its conduction band is rather small, so a non-negligible number of electrons are thermally excited to it, where they can move freely. The holes left behind in the valence band can also move.

Carbon atoms are smaller than Silicon atoms, so they are closer together, which makes the level splitting larger and thus the band gap larger for Carbon.

5. Lasers

A. (3 points) Define spontaneous emission, stimulated emission, and absorption of photons.

Spontaneous emission is when an atom in an excited state jumps to a lower energy state and emits a photon in a random direction.

Stimulated emission is when an atom in an excited state interacts with a photon whose energy equals the energy difference to some lower state. This causes it to jump to the lower energy state and emit a photon in exactly the same direction as the incident photon.

Absorption is when a photon whose energy equals the energy difference between an atom's initial state and some excited state interacts with the atom. The photon vanishes, and the atom is left in the excited state.

B. (3 points) Define population inversion, and explain why it is necessary for a laser to operate.

Population inversion is when an excited state has a higher population than a lower state. Because the cross section for stimulated emission is exactly the same as the cross section for absorption, if there are more atoms in the lower state, absortion will dominate over stimulated emission.

C. (3 points) In a 4-level laser system, what are the desirable relative lifetimes of the levels?

Atoms are somehow excited to level 4. We want a short lifetime for spontaneous decay from level 4 to level 3, and a long lifetime for spontaneous emission decay from level 3 to level 2. This will result is a high population in level 3. The laser transition is from level 3 to level 2 by stimulated emission. We want the lifetime for level 2 to decay to level 1 to be short. This keeps the population of level 2 small, so it's easier to maintain a population inversion between level 3 and level 2.

6. Rectangular Box Potential

A. (5 points) Write the time-independent Schrodinger equation in x, y, z coordinates for a potential $V(x, y, z)$.

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{xy^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V(x,y,z)\cdot\psi=E\psi$$

B. (5 points) A particle is in a box with $V=0$ for $0<x<w_x$ and $0<y<w_y$ and $0<z<w_z$ and $V=\infty$ elsewhere. Write the time-independent wavefunction for the particle. Define any variables you introduce. Don't worry about normalization.

The wavefunction is a 3D sinusoid inside the box, and zero at the walls.

The solution is $\psi(x,y,z)=\sin\left(\frac{n_x\pi x}{w_x}\right)\cdot\sin\left(\frac{n_y\pi y}{w_y}\right)\cdot\sin\left(\frac{n_z\pi y}{w_z}\right)$ with positive integer $n_x\ n_y\ n_z$

C. (5 points) Write the formula for the energies of the possible states for a particle of mass m in this potential.

The energies are $E_{nx,\,ny,\,nz}=\frac{\hbar^2\pi^2}{2m}\cdot\left[\frac{n_x^2}{w_x^2}+\frac{n_y^2}{w_y^2}+\frac{n_z^2}{w_z^2}\right]$

D. (5 points) If the dimensions are $w_x = w_y = 2$ nm and $w_z = 1$ nm, and the particles are electrons obeying the Pauli Exclusion Principle, what would be the total energy if there are 9 electrons in the box? Ignore repulsion between the electrons.

nx	ny	nz	[ ]
1	1	1	1.50
2	1	1	2.25
1	2	1	2.25
2	2	1	3.00
3	1	1	3.50
3	2	1	4.25
1	1	2	4.50

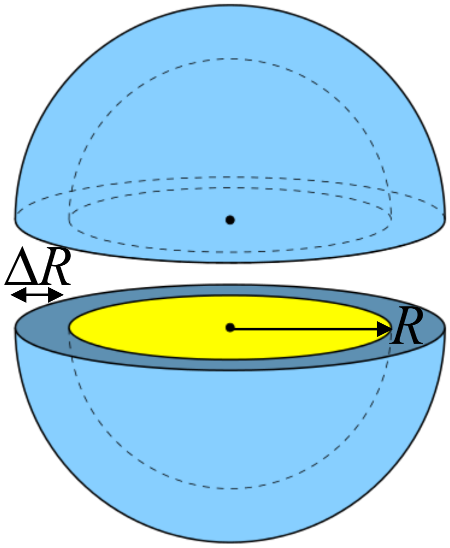
For those widths, $E_{nx,\,ny,\,nz}=\frac{\hbar^2\pi^2}{2m}\cdot\left[\frac{n_x^2}{4\text{ nm}^2}+\frac{n_y^2}{4\text{ nm}^2}+\frac{n_z^2}{1\text{ nm}^2}\right]$

For electrons, $\frac{\hbar^2\pi^2}{2m}=\frac{(\hbar c)^2\pi^2}{2mc^2}=\frac{(197.3\text{ eV-nm})^2\pi^2}{2\cdot 0.511\times 10^6\text{ eV}}=0.3759\text{ eV-nm}^2$,

so $E_{nx,\,ny,\,nz}=0.3759\text{ eV}\cdot\left[0.25\cdot n_x^2+0.25\cdot n_y^2+1\cdot n_z^2\right]$

The first 2 electrons go into 111, the next 4 go into 211 and 121, the next 2 go into 221.
That's 8 electrons. The 9th electron will go into 311 (or 131 with the same energy).
So $E=0.3759\text{ eV}\cdot\left[2\cdot(1.50+2.25+2.25+3.00)+3.50\right]=0.3759\text{ eV}\cdot\left[21.5\right]=8.082\text{ eV}$

7. Spherical Shell Potential



An electron is in a spherically symmetric shell potential that is infinite for $r < R$, and for $r > R+\Delta R$, and zero for $R < r < R+\Delta R$, with $R = 2$ nm and $\Delta R = 0.2$ nm.

A. (4 points) Write the reduced radial Schrodinger Equation that must be solved to find the energy levels.

$$-\frac{\hbar^2}{2m}\frac{\partial^2 U(r)}{\partial r^2}+\frac{\hbar^2\cdot\ell\cdot(\ell+1)}{2mr^2}\cdot U(r)=E\cdot U(r)$$

B. (4 points) What is the wavefunction for the ground state as a function of r, θ, ϕ? Don't worry about normalization.

The full wavefunction is $\psi_{k\ell m}=\frac{U_{k\ell}(r)}{r}\cdot Y_\ell^m(\theta,\phi)$

The ground state is $\ell=0, m=0$ and (un-normalized) $Y_0^0=1$.

The radial equation is just $-\frac{\hbar^2}{2m}\frac{\partial^2 U(r)}{\partial r^2}=E\cdot U(r)$ with $U(r) = 0$ at $r = R$ and $R+\Delta R$.

The $\ell=0$ solutions are $U_{k0}(r)=\sin\frac{k\pi\cdot(r-R)}{\Delta R}$ with positive integer k.

So the (un-normalized) ground state wavefunction is $\psi_{100}(r,\theta,\phi)=\frac{1}{r}\cdot\sin\frac{\pi\cdot(r-R)}{\Delta R}$

C. (4 points) What is the energy (include units) of the ground state?

The energy for an electron in an infinite square well of width ΔR is

$$E_k=k^2\cdot\frac{1}{2m}\cdot\left(\frac{\hbar\pi}{\Delta R}\right)^2=\frac{k^2}{\Delta R^2}\cdot\frac{(\hbar c\pi)^2}{2mc^2}=\frac{k^2}{\Delta R^2}\cdot\frac{(197.3\text{ eV-nm}\cdot\pi)^2}{2\cdot 0.511\times 10^6\text{ eV}}=\frac{k^2}{\Delta R^2}\cdot 0.3759\text{ eV-nm}^2$$

So $E_{10}=\frac{1^2}{(0.2\text{ nm})^2}\cdot 0.3759\text{ eV-nm}^2=9.398\text{ eV}$

D. (4 points) What is the approximate energy (include units) of the first excited state?

Since $R \gg \Delta R$, we approximate the extra term in the radial equation as a constant evaluated at $r = R + \frac{\Delta R}{2} = 2.1\text{ nm}$, which just shifts the energy up a bit.

The energy shift is $\Delta E_\ell = \frac{\hbar^2 \cdot \ell \cdot (\ell+1)}{2m \cdot \left(R + \frac{\Delta R}{2}\right)^2} = \frac{(\hbar c \pi)^2}{2mc^2} \cdot \frac{\ell \cdot (\ell+1)}{\pi^2 \cdot \left(R + \frac{\Delta R}{2}\right)^2}$

For this problem, $\Delta E_\ell = \frac{0.3759\text{ eV-nm}^2}{\pi^2 \cdot (2.1\text{ nm})^2} \cdot \ell \cdot (\ell+1) = .008636\text{ eV} \cdot \ell \cdot (\ell+1)$.

For $k = 1,\ \ell = 1,\ E = 9.398\text{ eV} + 0.008636\text{ eV} \cdot 2 = 9.415\text{ eV}$.

The $k = 2,\ \ell = 0$ energy $E_{20} = 2^2 \cdot E_{10} = 4 \cdot 9.398 = 37.592\text{ eV}$ is far higher

E. (4 points). What is the approximate total energy (include units) if there are 9 electrons in the potential obeying the Pauli exclusion principle? Ignore repulsion between the electrons.

The energy changes from increasing ℓ are much smaller than from increasing k, so the first 9 electrons will all have $k = 1$.

The first two electrons will be in the 1S state with $\ell = 0$.

The next electrons will be in the 1P state with $\ell = 1$. This has $m = -1$, $m = 0$, and $m = +1$ wavefunctions, each of which can hold 2 electrons, or 6 total. That gets us to 8 electrons.

The 9th electron will go into one of the many 1D states with $\ell = 2$, which all have the same energy.

The sum of the shifts is

$\Delta E = 0.008636\text{ eV} \cdot \{6 \cdot [1 \cdot (1+1)] + 1 \cdot [2 \cdot (2+1)]\} = 0.008636\text{ eV} \cdot \{18\} = 0.1554\text{ eV}$

so the total energy is $E = 9.398\text{ eV} \cdot 9 + 0.1554\text{ eV} = 84.7374\text{ eV}$

Putting electrons only in $\ell = 0$ states with $k = 1, 2, 3, 4, 5$ is wrong, but gives

$2 \cdot E_{10} + 2 \cdot E_{20} + 2 \cdot E_{30} + 2 \cdot E_{40} + E_{50} = [2 \cdot (1+4+9+16) + 25] \cdot E_{10} = 85 \cdot 9.398 = 798.83\text{ eV}$