PHYS 250 Midterm Solutions
June 13, 2023

1. A muon with relativistic energy 5 GeV is produced at 50 km altitude, aimed at the ground. Muons have mass 105.7 MeV/c² and lifetime $\tau = 2.2$ μs in their own rest frame.
The number of muons remaining at time t is $N(t) = N_{t=0} \cdot e^{-t/\tau}$

A. What is the gamma of the muon? (8 points)

$$\gamma = \frac{E}{mc^2} = \frac{5\times10^9 \text{ eV}}{105.7\times10^6 \text{ eV/c}^2 \cdot c^2} = 47.30$$

B. What is the lifetime of the muon in the Earth frame? (8 points)

$$\tau_{\text{Earth}} = \gamma \cdot \tau_{\text{rest}} = 47.30 \cdot 2.2\,\mu\text{s} = 104.1\ \mu\text{s}$$

C. What is the distance to the Earth in the muon frame? (8 points)

$$D_{\text{muon}} = \frac{D_{\text{Earth}}}{\gamma} = \frac{50 \text{ km}}{47.30} = 1.057 \text{ km}$$

D. How long does it take to hit the Earth in the muon frame ? (8 points)

Since $E >> mc^2$, the muon is moving at essentially the speed of light.

$$t_{\text{muon}} = \frac{D_{\text{muon}}}{c} = \frac{1.057\times10^3 \text{ m}}{3\times10^8 \text{ m/s}} = 3.523\times10^{-6} \text{ s} = 3.523\ \mu\text{s}$$

E. What is the probability that the muon hits the Earth before decaying? (8 points)

$$P = e^{-\frac{t_{\text{muon}}}{\tau}} = e^{-\frac{3.523\,\mu s}{2.2\,\mu s}} = 0.2016$$

2. An X-ray tube has voltage of 100 kV.

A. What is the shortest X-ray wavelength produced? (10 points)

$$\lambda = \frac{h}{E} = \frac{hc}{E} = \frac{1240 \text{ eV-nm}}{100 \times 10^3 \text{ eV}} = 1.240 \times 10^{-2} \text{ nm} = 12.40 \text{ pm}$$

B. What is the minimum angle for Bragg diffraction of the X-rays from a crystal with layer spacing of 80 pm? State the units you are using. (10 points)

$$2d \sin\theta_{\text{surface}} = n\lambda \rightarrow \sin\theta_{\text{min}} = \frac{1 \cdot \lambda}{2d} = \frac{12.40 \text{ pm}}{2 \cdot 80 \text{ pm}} = 7.750 \times 10^{-2}$$
$$\theta_{\text{min}} = \sin^{-1}\left(7.750 \times 10^{-2}\right) = 77.58 \text{ mrad} = 4.445°$$

C. If a minimum-wavelength X-ray is Compton-scattered through 120°,
what is the energy of the scattered photon? (10 points)

$$\lambda' - \lambda = \frac{hc}{m} \cdot \left(1 - \cos\theta\right) = 2.426 \text{ pm} \cdot \left(1 - \cos 120°\right) = 2.426 \cdot \left(1 - \left(-0.5\right)\right) = 3.639 \text{ pm}$$
$$\lambda' = 12.40 \text{ pm} + 3.639 \text{ pm} = 16.04 \text{ pm}$$
$$E' = \frac{hc}{\lambda'} = \frac{1240 \text{ eV-nm}}{16.04 \text{ pm}} = 7.731 \times 10^4 \text{ eV} = 77.31 \text{ keV}$$

D. If the X-ray tube anode is made of tungsten (Z = 74),
what is the wavelength of the Kα (*n* = 2 to *n* = 1 transition) X-ray peak? (10 points)

$$E_{K\alpha} = 13.6 \text{ eV} \cdot \left(\frac{1}{1^2} - \frac{1}{2^2}\right) \cdot \left(Z - 1\right)^2 = 13.6 \cdot 0.75 \cdot \left(74 - 1\right)^2 = 54.36 \text{ keV}$$
$$\lambda = \frac{hc}{E} = \frac{1240 \text{ eV-nm}}{54.36 \times 10^3 \text{ eV}} = 2.281 \times 10^{-2} \text{ nm} = 22.81 \text{ pm}$$

3. Atoms

A. The green colour of an aurora borealis is due to transitions from $n = 4$ to $n = 3$ of an electron orbiting a core of doubly-ionized oxygen with an effective $Z = 2$. What is the photon wavelength in the Bohr model? (10 points)

$$E = 13.6\ \text{eV} \cdot Z^2 \cdot \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) = 13.6 \cdot 2^2 \cdot \left(\frac{1}{3^2} - \frac{1}{4^2} \right) = 2.644\ \text{eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240\ \text{eV-nm}}{2.644\ \text{eV}} = 468.9\ \text{nm}$$

B. Show that requiring that the circumference of a circular orbit with radius r to be an integer number n of de Broglie wavelengths results in quantization of angular momentum: $L = r \times p = n\hbar$ (10 points)

$$2\pi r = n\lambda_{\text{de Broglie}} = n\frac{h}{p} \;\rightarrow\; r \times p = n\frac{h}{2\pi} \;\rightarrow\; L = n\frac{h}{2\pi} = n\hbar$$

C. The maximum possible resolution of a microscope is approximately the wavelength used, so for a light microscope, the maximum possible resolution is about 500 nm. What is the maximum possible resolution of an electron microscope using 10 keV electrons? (10 points)

[Note that achieving the maximum resolution requires impractically large lens apertures, and optically perfect lenses, but we are ignoring that here.]

$$E = \frac{p^2}{2m} \rightarrow p = \sqrt{2mE}$$

$$\lambda_{\text{de Broglie}} = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

$$\lambda_{\text{electron}} = \frac{hc}{\sqrt{2m_e c^2 E_{\text{electron}}}} = \frac{1240\ \text{eV-nm}}{\sqrt{2 \cdot 0.511 \times 10^6\ \text{eV} \cdot E_{\text{electron}}}} = \frac{1.227\ \sqrt{\text{eV}} \cdot \text{nm}}{\sqrt{E_{\text{electron}}}}$$

$$\lambda_{\text{electron}} = \frac{1.227\ \sqrt{\text{eV}} \cdot \text{nm}}{\sqrt{10\ \text{keV}}} = 1.227 \times 10^{-2}\ \text{nm} = 12.27\ \text{pm}$$