PHYS 250

Lecture 1.2

More Time-Space Relativity

Today

Administrivia

Review & Leftovers

More on Beta and Gamma

Relativistic Velocity Addition

Relativistic Doppler Effect

Momentum, Energy, and Mass in Relativity

Schedule

Midterm: Tuesday June 3, 11:30-12:30. (to be confirmed)

Office Hours: Friday 2:30-3:30 (and later at Koerners….)

Tutorials: Friday 8:30-10

No class next Monday

	Monday	Tuesday	Wednesday	Thursday	Friday
8:30	P250		P250		P250
9:00	P250		P250		P250
9:30	P250		P250		P250
10:00	P250		P250		E270
10:30	M257		M257		E270
11:00	M257		M257		E270
11:30	M257	midterm	M257		E270
12:00	M257	midterm	M257		
12:30		M257		M257	E257
13:00	E270	M257	E270	M257	E257
13:30	E270	M257	E270	M257	E257
14:00	E270	E253	E270	E253	
14:30	E270	E253	E270	E253	Office Hours
15:00	E257	E253	E257	E253	Office Hours
15:30	E257	E253	E257	E253	
16:00	E257	E253	E257	E253	Office Hours
16:30		E253		E253	Office Hours
17:00		E253		E253	Office Hours
17:30		E253		E253	Office Hours
18:00					
18:30					

Review

The speed of light doesn't depend on the motion of an observer.

Things that are simultaneous to one observer may not be simultaneous to another.

A clock that is regulated by light pulses appears to slow down when moving.

The transformation laws required for the above to be true are

$$x' = \frac{x - ut}{\sqrt{1 - \left(u/c\right)^2}} \qquad t' = \frac{t - ux/c^2}{\sqrt{1 - \left(u/c\right)^2}}$$

where u is the velocity in the x-direction of the S' frame relative to the S frame and c is the speed of light.

Simplifying the Notation

Define $\beta = \frac{u}{c}$ and $\gamma = \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$.

Then we can write: $x' = \gamma \cdot (x - ut)$ $\quad t' = \gamma \cdot \left(t - \frac{u}{c}\frac{x}{c} \right) = \gamma \cdot \left(t - \beta \frac{x}{c} \right)$

Instead of *t* and *t'*, write *ct* and *ct'*:

$$x' = \gamma \cdot \left(x - \frac{u}{c} \cdot ct \right) = \gamma \cdot (x - \beta ct) \qquad ct' = \gamma \cdot (ct - \beta x)$$

Note the similarity of the above formulas: $x \leftrightarrow ct \quad x' \leftrightarrow ct'$

Review 2

A moving clock appears to slow down by a factor of γ.

The length of a moving object is defined by the distance between the ends, measured at the same time, using the time and space coordinates of one observer.

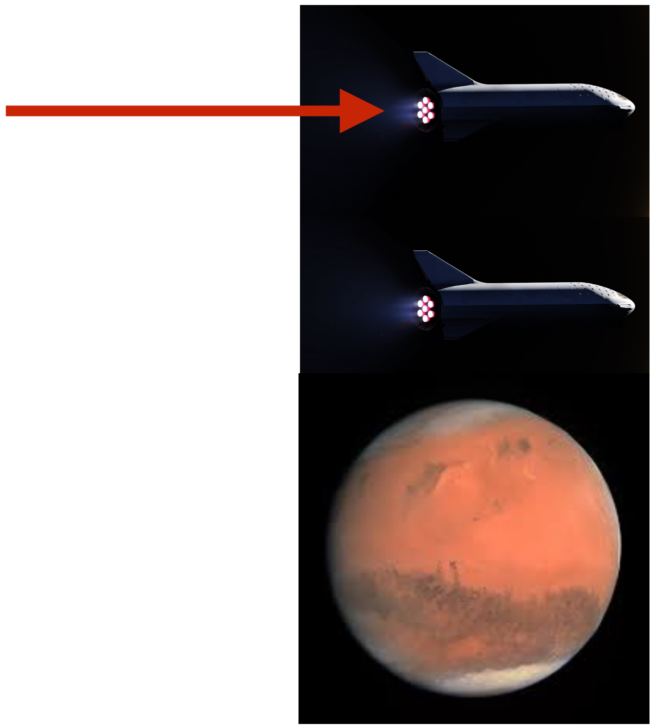
The length of a moving object appears to be shorter by a factor of γ.

Elon-gth X

A SpaceX Starship with length 50 meters passes by Mars at $\beta = 0.2$.

Compared to an identical Starship orbiting Mars, the moving spaceship appears

A. Longer
B. The same length
<u>C. Shorter</u>
D. I don't know how to do this

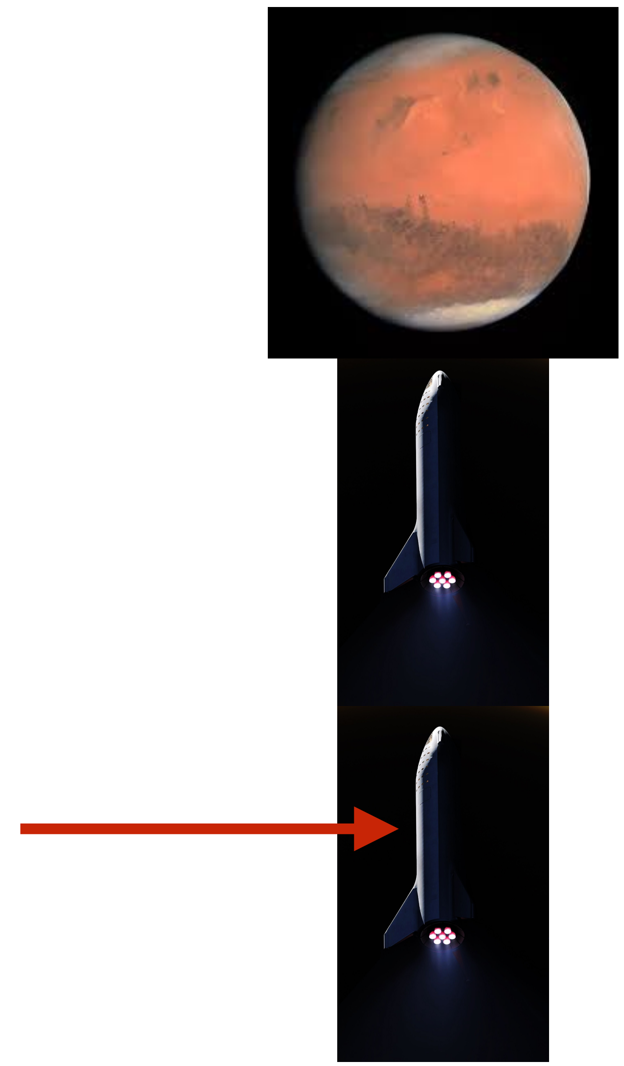


Although it's only 2% shorter
despite moving at 60,000 km per second !

Elon-gth Y

The same Starship at the same velocity $\beta = 0.2$ rotates 90° before passing Mars. Compared to an identical Starship orbiting Mars, the moving spaceship appears

A. Longer
<u>B. The same length</u>
C. Shorter
D. I don’t know how to do this



Transverse Transformations

Frame S' has velocity u in the $+x$ direction as seen from S, whose origins overlap at $t = t' = 0$.

A light pulse is emitted in the y direction at $x_1 = y_1 = t_1 = 0$ in S.
In S', that is $x_1' = y_1' = t_1' = 0$.

The light pulse is absorbed at $x_2 = 0$, $y_2 = cT$, $t_2 = T$ in S.
We know how to find x_2' and t_2'

$$x_2' = \gamma\left(x_2 + \beta c t_2\right) = \gamma\left(0 + \beta cT\right) = \beta\gamma cT$$
$$ct_2' = \gamma\left(ct_2 + \beta x_2\right) = \gamma\left(cT + \beta 0\right) = \gamma cT$$

We need to find/guess the transformation law for y_2'
that will give the right speed of light in S'.

Transverse Transformations 2

We want $\dfrac{\Delta s}{\Delta t} = \dfrac{\sqrt{\left(x_2' - x_1'\right)^2 + \left(y_2' - y_1'\right)^2}}{t_2' - t_1'} = c$

Rearrange and square: $\left(x_2' - x_1'\right)^2 + \left(y_2' - y_1'\right)^2 = \left(ct_2' - ct_1'\right)^2$

Plug in: $\left(\beta\gamma cT - 0\right)^2 + \left(y_2' - 0_1\right)^2 = \left(\gamma cT - 0\right)^2$

Simplify: $\left(\beta\gamma cT\right)^2 + \left(y_2'\right)^2 = \left(\gamma cT\right)^2$

Find y_2’: $\left(y_2'\right)^2 = \left(\gamma^2 - \beta^2\gamma^2\right)\left(cT\right)^2$

Expand: $\gamma^2 - \beta^2\gamma^2 = \gamma^2\left(1-\beta^2\right) = \left(\dfrac{1}{\sqrt{1-\beta^2}}\right)^2\left(1-\beta^2\right) = \dfrac{1-\beta^2}{1-\beta^2} = 1$

So $\left(y_2'\right)^2 = 1\cdot\left(cT\right)^2 = \left(y_2\right)^2$. The *y* coordinate doesn’t change!

3D Lorentz Transformations

The same argument works for z, or any combination of y and z.

$$ct' = \gamma\left(ct - \beta x\right)$$
$$x' = \gamma\left(x - \beta ct\right)$$
$$y' = y$$
$$z' = z$$

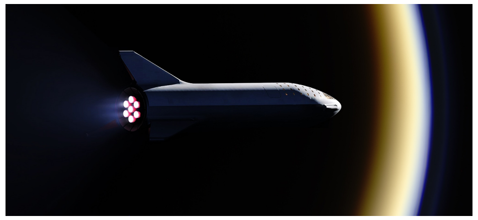
$$\beta = \frac{u}{c}$$ u along the x direction

$$\gamma = \frac{1}{\sqrt{1-\left(u/c\right)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

Starships Passing in the Night

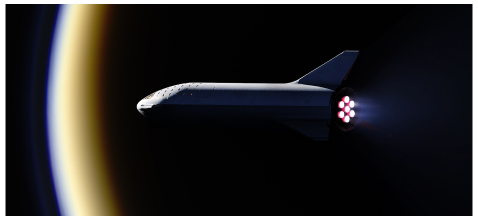
Starship A passes Mars with $\beta = +2/3$

A



while Starship B passes Mars with $\beta = -2/3$

B

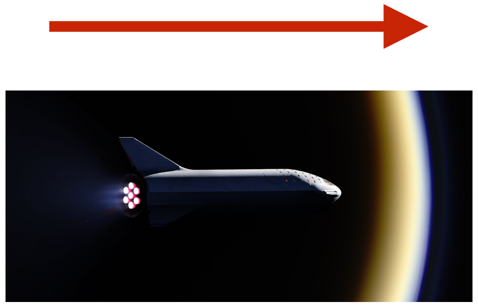


A. A sees B's clock tick slower than his own,
   B sees A's clock tick faster than his own
B. B sees A's clock tick slow than his own,
   A sees B's clock tick faster than his own
C. Both see the other's clock tick slower than their own
D. Both see the other's clock to have stopped ticking
E. Neither Starship can even see the other after they pass Mars

Starships Passing in the Night

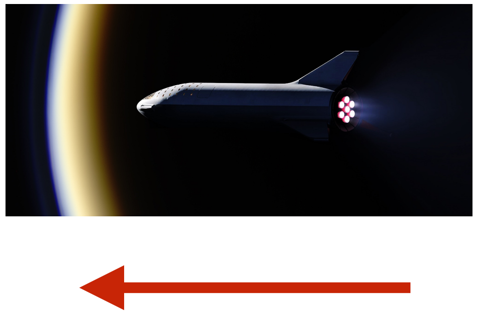
Starship A passes Mars with $\beta = +2/3$

A



while Starship B passes Mars with $\beta = -2/3$

B



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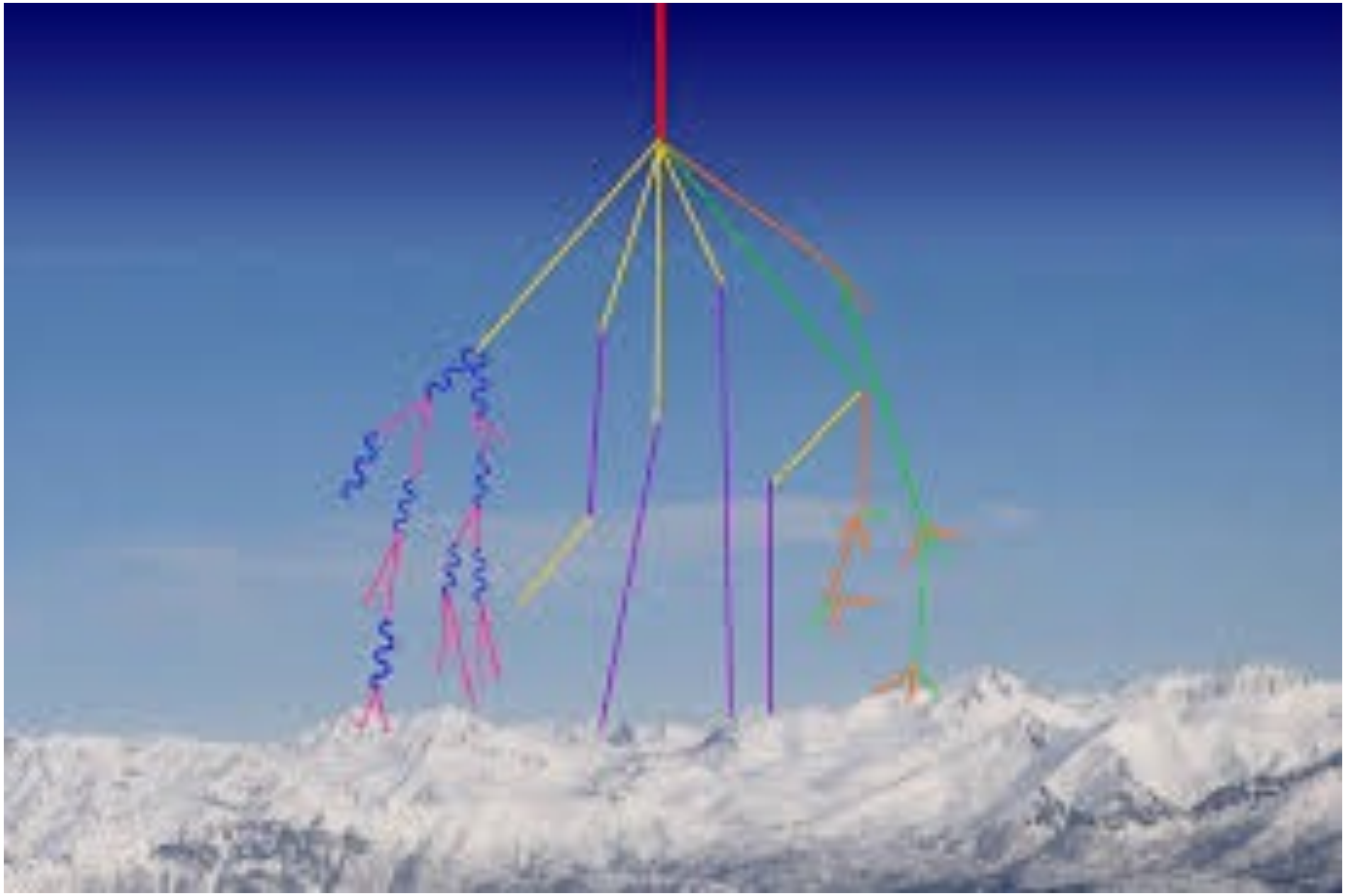
<u>C. Both see the other's clock tick slower than their own</u>

D. Both see the other's clock to have stopped ticking

E. Neither Starship can even see the other after they pass Mars

Lorentz Contraction and Time Dilation Are Real

Cosmic rays are high energy particles from space. In space, they are mostly protons, with some atomic nuclei. They interact with air nuclei at altitudes 10-100 km, making other kinds of particles. At sea level, almost all charged particles you see are electrons (or positrons) and muons.

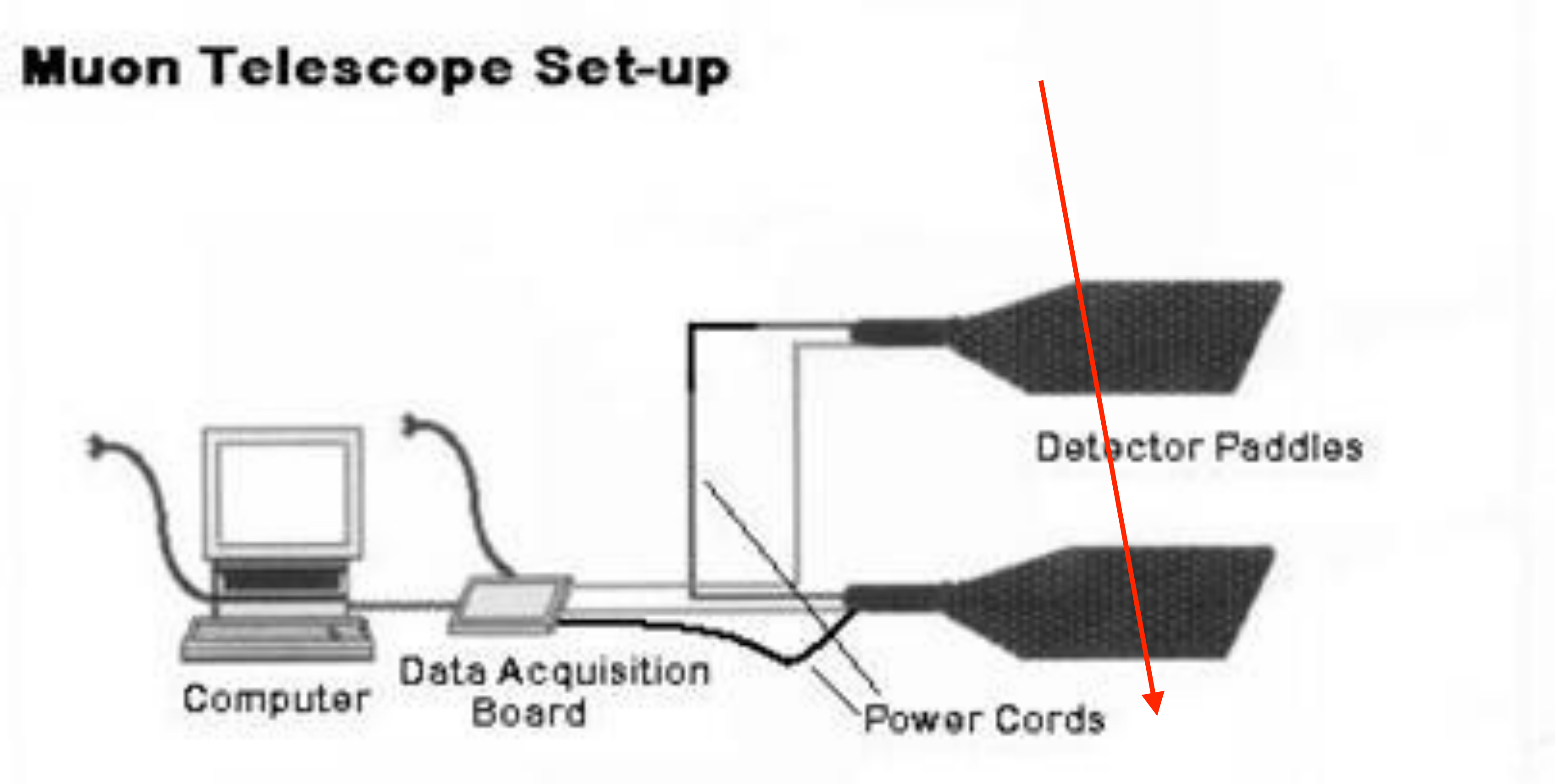


Cosmic Ray Particle Speeds

Some plastics give a light flash when a particle goes through, and a photomultiplier tube turns the light into a narrow electrical pulse. The time accuracy is better than a nanosecond.

You can have two or more such detectors separated by a few meters.

You never see any particles go faster than the speed of light, and right at the speed of light is the most common.



Stopped Muon Lifetime

Sometimes a particle called a muon will go through the top scintillator and stop in the bottom one (especially if it's thick).

A muon decays into an electron and neutrinos, and the electron makes a second flash in the scintillator. The exponential lifetime of a muon is ~2200 nanoseconds.

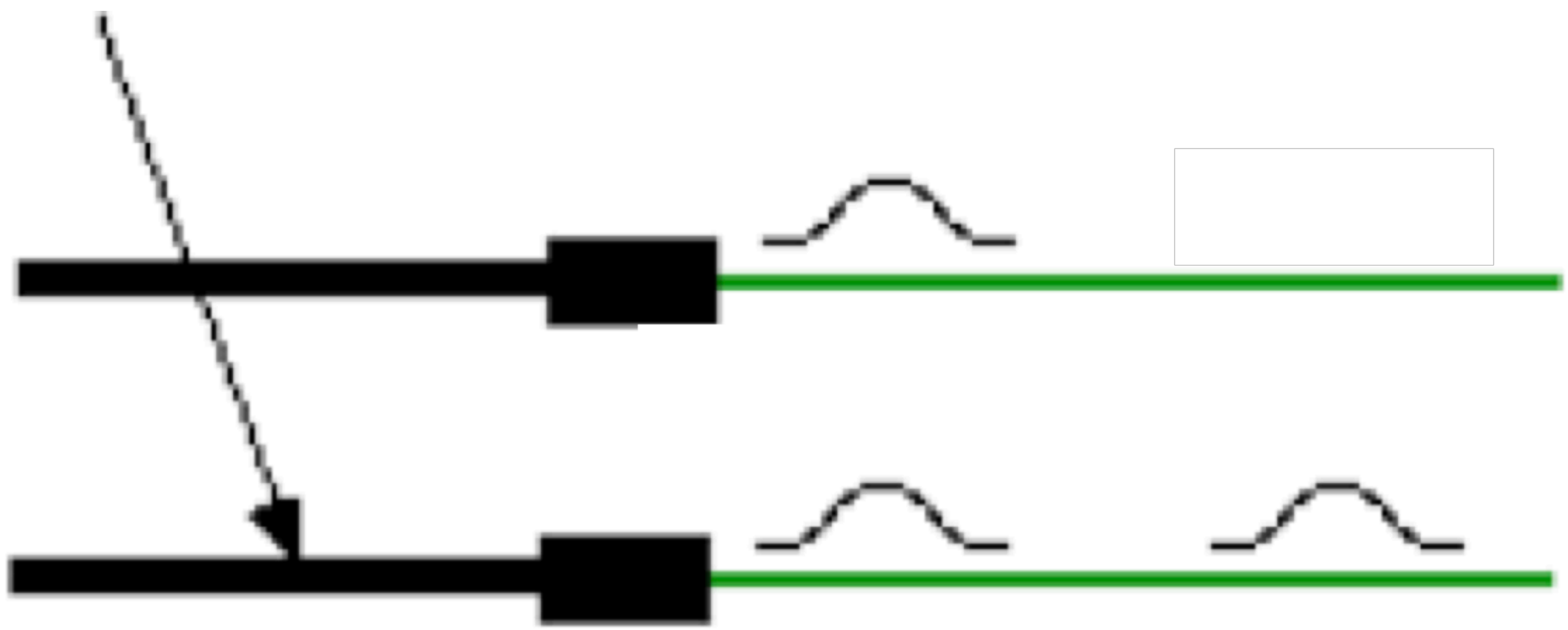


Figure 3: Time delay between the two pulses from the lower scintillator shows the time for the muon to decay.

Why Are There Any Muons at Sea Level?

A particle at the speed of light travels 660 meters in 2200 nanoseconds.

Since the muons are created at altitudes of more than 10 km, and more like 50 km, there shouldn't be any of them reaching the ground. But there are plenty !

The reason is relativistic time dilation.

The muons are created with velocities very close to the speed of light.
This makes them live long enough to reach the ground.

What About the Muon's Frame?

In the muon's own frame, it sees the ground coming up to meet it at the speed of light. If the ground is 50 km away, it will take 1.66×10^5 nanoseconds to get to the muon.

In its own rest frame, the muon lives ~ 2200 nanoseconds.

So all the muons should decay before the ground hits them.

In this case, it's the length contraction that fixes things.

The muon sees the 50 km as a much shorter distance, so the ground can get to it before the muon decays.

Beta and Gamma

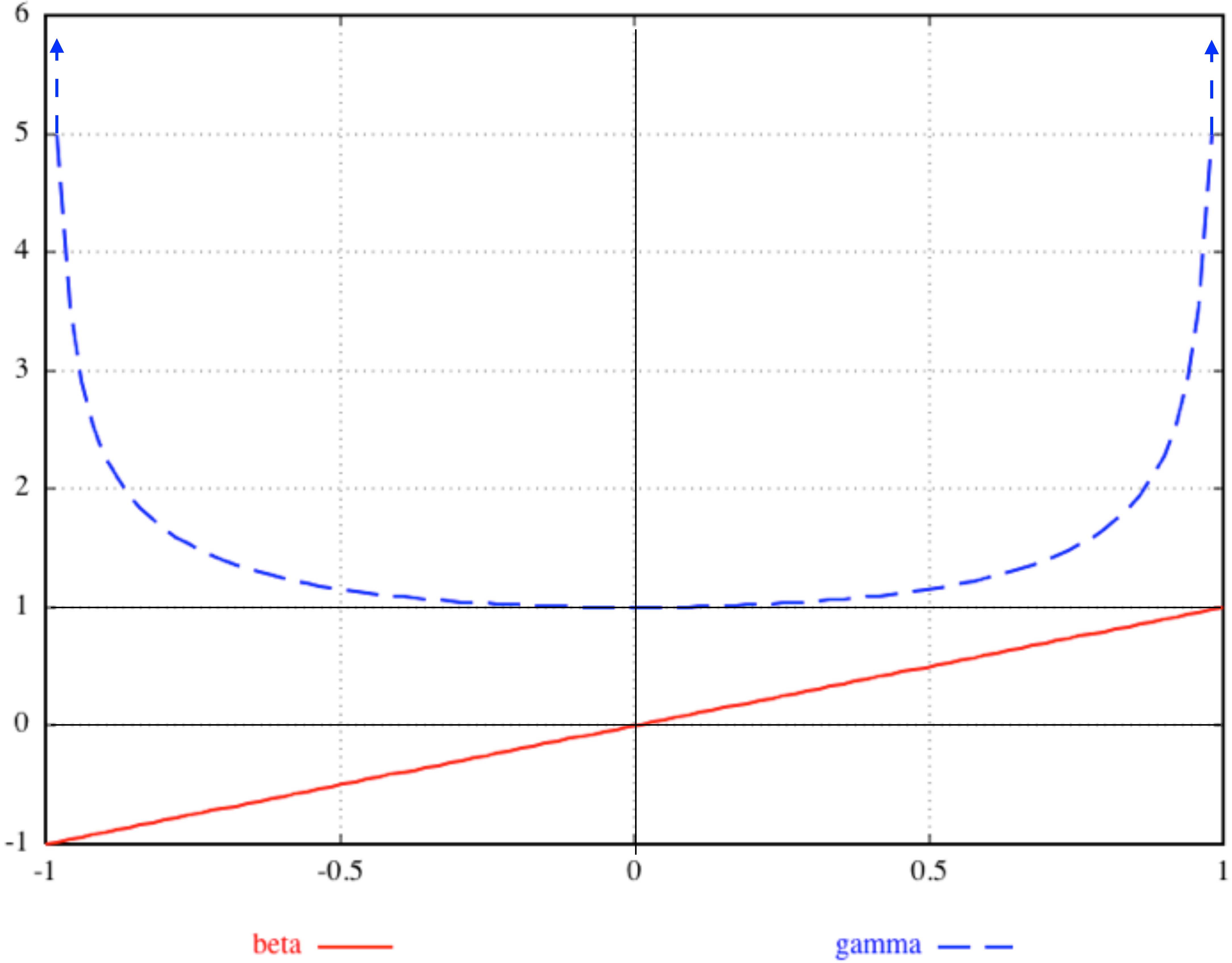
$$\beta = \frac{u}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{Inverting, } \gamma^2 = \frac{1}{1-\beta^2} \rightarrow 1-\beta^2 = \frac{1}{\gamma^2} \rightarrow \beta = \sqrt{1-1/\gamma^2}$$

$\beta = 0$ corresponds to rest, increasing β means faster.

$\beta = 1$ is the speed of light. It will turn out that $|u| \leq c$ so $|\beta| \leq 1$.

$\gamma = 1$ corresponds to rest, increasing γ means faster. γ is never less than 1.

Beta and Gamma vs *u*/*c*



Beware of Calculator Precision !

How much slower does a clock run that is moving at 1 meter per second?

$$\beta = \frac{1\text{ m/s}}{3\times 10^{8}\text{ m/s}} = 3.33\times 10^{-9} \quad \gamma = \frac{1}{\sqrt{1-\left(3.33\times 10^{-9}\right)^{2}}} = 1.0000$$

Not enough decimal places. So subtract 1. The answer is 0.0000.

There are very few calculators that can do this correctly. (Wolfram can).

Taylor-expanding gives $\gamma \approx 1+\frac{1}{2}\beta^{2} = 1+\frac{\left(3.33\times 10^{-9}\right)^{2}}{2} = 1+5.56\times 10^{-18}$.

Any scientific calculator can do that.

Beware of Calculator Precision 2

A photon and electron with energy 5.11 TeV ($\gamma = 10^7$) are emitted simultaneously
After travelling for 1 microsecond, how far behind the photon is the electron?

The photon has $v = c$. We need the β of the electron to find its velocity.

Plug into the exact formula $\beta = \sqrt{1 - 1/\gamma^2} = \sqrt{1 - 1/\left(10^7\right)^2} = 1.0000$

Subtract 1, and the answer is 0.0000. Calculator precision again.

For large γ, Taylor-expanding gives $\beta \approx 1 - \frac{1}{2}\frac{1}{\gamma^2}$ or $1 - \beta \approx \frac{1}{2\gamma^2}$

So $1 - \beta \approx \frac{1}{2 \cdot \left(10^7\right)^2} = 5.0 \times 10^{-15}$. Times $c = 3 \times 10^8$ m/s gives $\Delta v = 1.5 \times 10^{-6}$ m/s

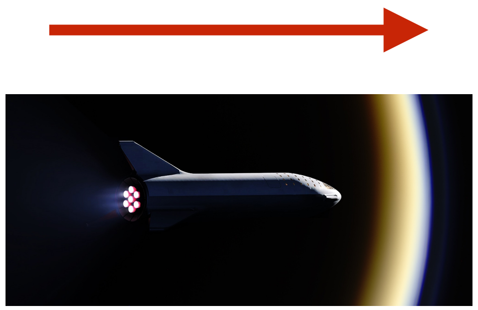
Times $t = 10^{-6}$ seconds gives $\Delta x = 1.5 \times 10^{-12}$ meters behind the photon.

1.5 picometers.

Starships Passing in the Night 2

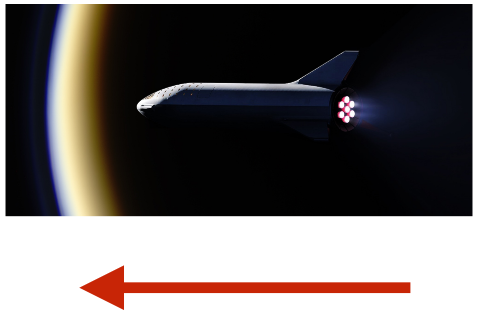
Starship A passes Mars with $\beta = +2/3$

A



while Starship B passes Mars with $\beta = -2/3$

B



What is the motion of A as observed by B?

A. $\beta = +\,4/3$
B. $\beta = -\,4/3$
C. It's too early for me to do this
D. Even if it weren't too early, I don't know how to do this

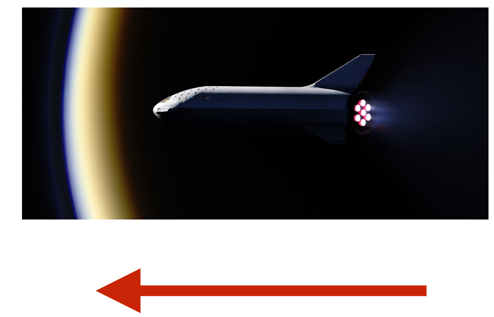
Starships Passing in the Night 2

Starship A passes Mars with $\beta = +2/3$

A

while Starship B passes Mars with $\beta = -2/3$

B



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A. $\beta = +\,4/3$
B. $\beta = -\,4/3$
C. It's too early for me to do this
D. Even if it weren't too early, I don't know how to do this

Relativistic Velocity Addition

If an object is moving with velocity *v'* as measured in *S'*, which is moving with velocity *u* seen from the *S* frame, what is the object's velocity in the *S* frame?

Let's say the object was at $x = x' = 0$ at time $t = t' = 0$.
In the *S'* frame, it will be at $x' = v't'$ at time t'.
In the *S* frame, that corresponds to

$$x = \gamma\left(x' + \beta ct'\right) = \gamma\left(v't' + \frac{u}{c}ct'\right) = \gamma\left(v' + u\right)t'$$

$$ct = \gamma\left(ct' + \beta x'\right) = \gamma\left(ct' + \frac{u}{c}v't'\right) = \gamma\left(1 + \frac{uv'}{c^2}\right)ct'$$

The velocity in the *S* frame is

$$v = \frac{x}{t} = \frac{x}{ct\frac{1}{c}} = \frac{\gamma\left(v' + u\right)t'}{\gamma\left(1 + \frac{uv'}{c^2}\right)ct'\frac{1}{c}} = \frac{v' + u}{1 + \frac{v' \cdot u}{c^2}}$$

This is relativistic velocity addition (for *v*, *v'*, and *u* in the *x* direction).

Relativistic Beta Addition

The formula is less messy using β values instead of velocities.

Take $v = \dfrac{v' + u}{1 + \dfrac{v' \cdot u}{c^2}}$, factor the denominator, divide by c, giving $\dfrac{v}{c} = \dfrac{v'/c + u/c}{1 + \dfrac{v'}{c} \cdot \dfrac{u}{c}}$.

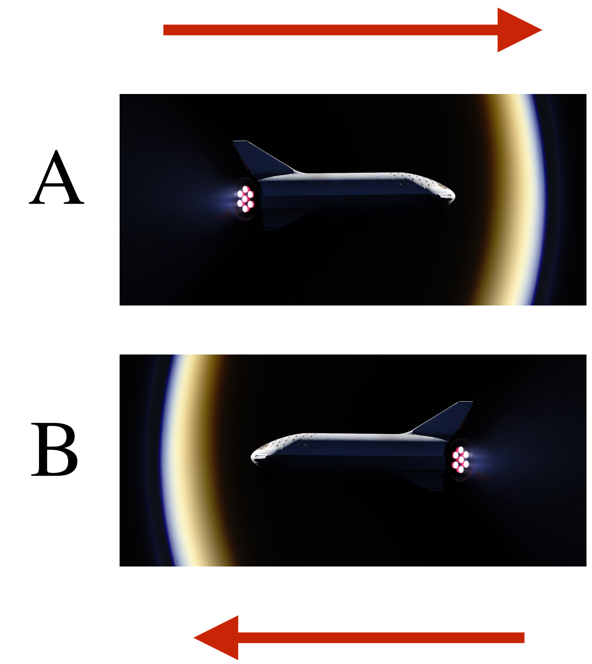
$\dfrac{v'}{c}$ is the motion of the object in the S' frame. Call it β_1.

$\dfrac{u}{c}$ is the motion of the S' frame itself in the S frame. Call it β_2.

$\dfrac{v}{c}$ is the motion of the object in the S frame. Call it β_{1+2}.

Then $\beta_{1+2} = \dfrac{\beta_1 + \beta_2}{1 + \beta_1 \cdot \beta_2}$. Much nicer!

Starships Passing in the Night 3



Starship A passes Mars with $\beta = +2/3$
while Starship B passes Mars with $\beta = -2/3$
What is the velocity of A as observed by B?

In the Mars frame, A has $\beta_1 = +2/3$.

In the B frame, the Mars frame is moving to the right with $\beta_2 = +2/3$

So B sees the motion of A to be $\beta_{1+2} = \dfrac{(+2/3)+(+2/3)}{1+(+2/3)\cdot(+2/3)} = \dfrac{+4/3}{1+4/9} = +0.9231$

The motions don't add up to $\beta = +4/3$.

They add up to a bit less than the speed of light.

Classical Doppler Effect

For an <u>observer at rest</u> in a medium and a <u>source moving away</u> at velocity u, the observers sees a wave period of

$$\Delta T_{\text{obs}} = \Delta T_{\text{source}} + \frac{u\Delta T_{\text{source}}}{c} = \Delta T_{\text{source}} \cdot \left(1 + u/c\right)$$

Since frequency $f = 1/\Delta T$, $f_{\text{obs}} = \dfrac{f_{\text{source}}}{1 + u/c} \approx f_{\text{source}} \cdot \left(1 - u/c\right)$

Since wavelength $\lambda = c\Delta T$, $\lambda_{\text{obs}} = \lambda_{\text{source}} \cdot \left(1 + u/c\right)$

For a <u>source at rest</u> in a medium, and an <u>observer moving away</u> at velocity u,

$$\Delta T_{\text{obs}} = \Delta T_{\text{source}} + \frac{u\Delta T_{\text{obs}}}{c} \Rightarrow \Delta T_{\text{obs}} = \frac{\Delta T_{\text{source}}}{1 - u/c} \approx \Delta T_{\text{source}} \cdot \left(1 + u/c\right)$$

Take reciprocals: $f_{\text{obs}} = f_{\text{source}} \cdot \left(1 - u/c\right)$

Multiply by c: $\lambda_{\text{obs}} = \dfrac{\lambda_{\text{source}}}{1 - u/c} \approx \lambda_{\text{source}} \cdot \left(1 + u/c\right)$

Moving observer and moving force are different for u comparable (or greater) than c. But for u << c, they are approximately the same.

Extreme Classical Doppler Effect

For an observer at rest, $f_{\text{obs}} = \dfrac{f_{\text{source}}}{1+u/c}$.

We get a finite (but very low) frequency for source velocity $u > c$.

If the source is approaching, we have $u < 0$.
The observed frequency goes to infinity as $u \to -c$.
We get a nonsense negative frequency for $u < -c$.
The source is moving faster than its waves, so we can't even see it.

For a source at rest, $f_{\text{obs}} = f_{\text{source}}\left(1-u/c\right)$.

The frequency goes to zero as observer velocity $u \to c$.
We get nonsense negative frequency for $u > c$, because the observer is out-running the waves. For negative velocity, the observed frequency is higher, and we get a finite (very high) frequency for $u < -c$.

Relativistic Doppler Effect

From the observer's point of view, the clock in a moving source runs slower, so the time between wavefront emissions is increased by a factor of γ.

For an <u>observer at rest</u> and a <u>source moving away</u>, using the observer's clock,

$$\Delta T_{\text{obs}} = \gamma\ \Delta T_{\text{source}} \left(1 + \frac{u}{c}\right) = \Delta T_{\text{source}} \frac{1 + u/c}{\sqrt{1 - u^2/c^2}}$$

$$= \Delta T_{\text{source}} \frac{1 + u/c}{\sqrt{(1 + u/c)(1 - u/c)}} = \Delta T_{\text{source}} \frac{\sqrt{1 + u/c}}{\sqrt{1 - u/c}}$$

For frequency, it's the reciprocal: $f_{\text{obs}} = f_{\text{source}} \sqrt{\dfrac{1 - u/c}{1 + u/c}}$

For wavelength, multiply both sides by c: $\lambda_{\text{obs}} = \lambda_{\text{source}} \sqrt{\dfrac{1 + u/c}{1 - u/c}}$

Relativistic Doppler Effect 2

For a source at rest, and an observer moving away, we use the classical wave-front timing result, but the gamma factor goes with the moving observer.

$$\gamma\ \Delta T_{\text{obs}} = \frac{\Delta T_{\text{source}}}{1-u/c} \Rightarrow \Delta T_{\text{obs}} = \Delta T_{\text{source}} \cdot \frac{\sqrt{1-u^2/c^2}}{1-u/c} = \Delta T_{\text{source}} \cdot \sqrt{\frac{1+u/c}{1-u/c}}$$

For frequency, it's the reciprocal: $f_{\text{obs}} = f_{\text{source}} \cdot \sqrt{\frac{1-u/c}{1+u/c}}$

For wavelength, multiply both sides by c: $\lambda_{\text{obs}} = \lambda_{\text{source}} \cdot \sqrt{\frac{1+u/c}{1-u/c}}$

Note that unlike the classical medium case, we get the same result whether the source is moving or the observer is moving.

Beware of differing sign conventions in these formulas.
My convention: u is positive if the source and observer are separating.

Also be careful about period vs wavelength vs frequency.

Relativistic Doppler Effect 3

Again, we get nicer looking formulas using β instead of velocity.

$$\Delta T_{\text{obs}} = \Delta T_{\text{source}} \cdot \sqrt{\frac{1+u/c}{1-u/c}} \rightarrow \Delta T_{\text{source}} \cdot \sqrt{\frac{1+\beta}{1-\beta}}$$

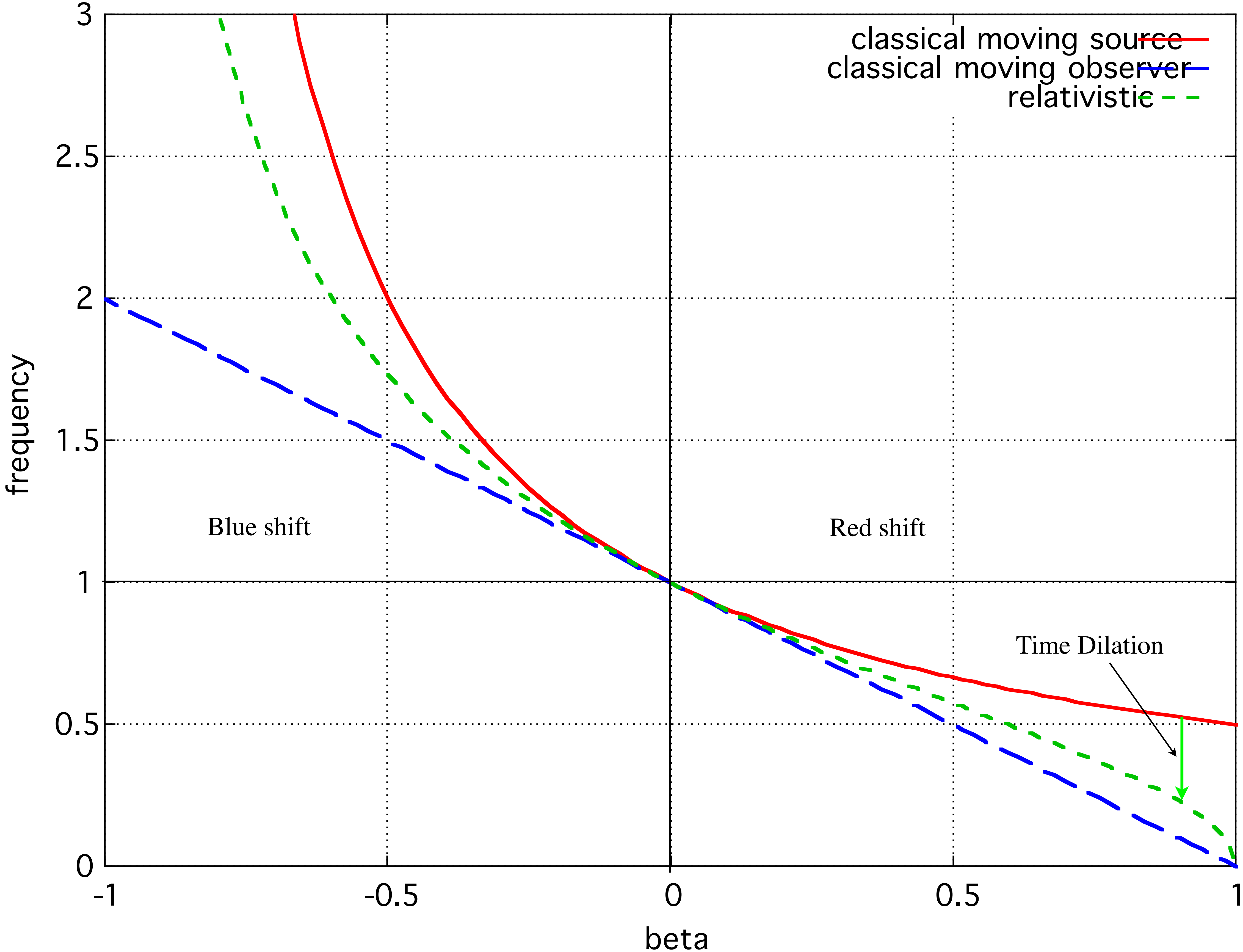
$$f_{\text{obs}} = f_{\text{source}} \cdot \sqrt{\frac{1-u/c}{1+u/c}} \rightarrow f_{\text{source}} \cdot \sqrt{\frac{1-\beta}{1+\beta}}$$

$$\lambda_{\text{obs}} = \lambda_{\text{source}} \cdot \sqrt{\frac{1+u/c}{1-u/c}} \rightarrow \lambda_{\text{source}} \cdot \sqrt{\frac{1+\beta}{1-\beta}}$$

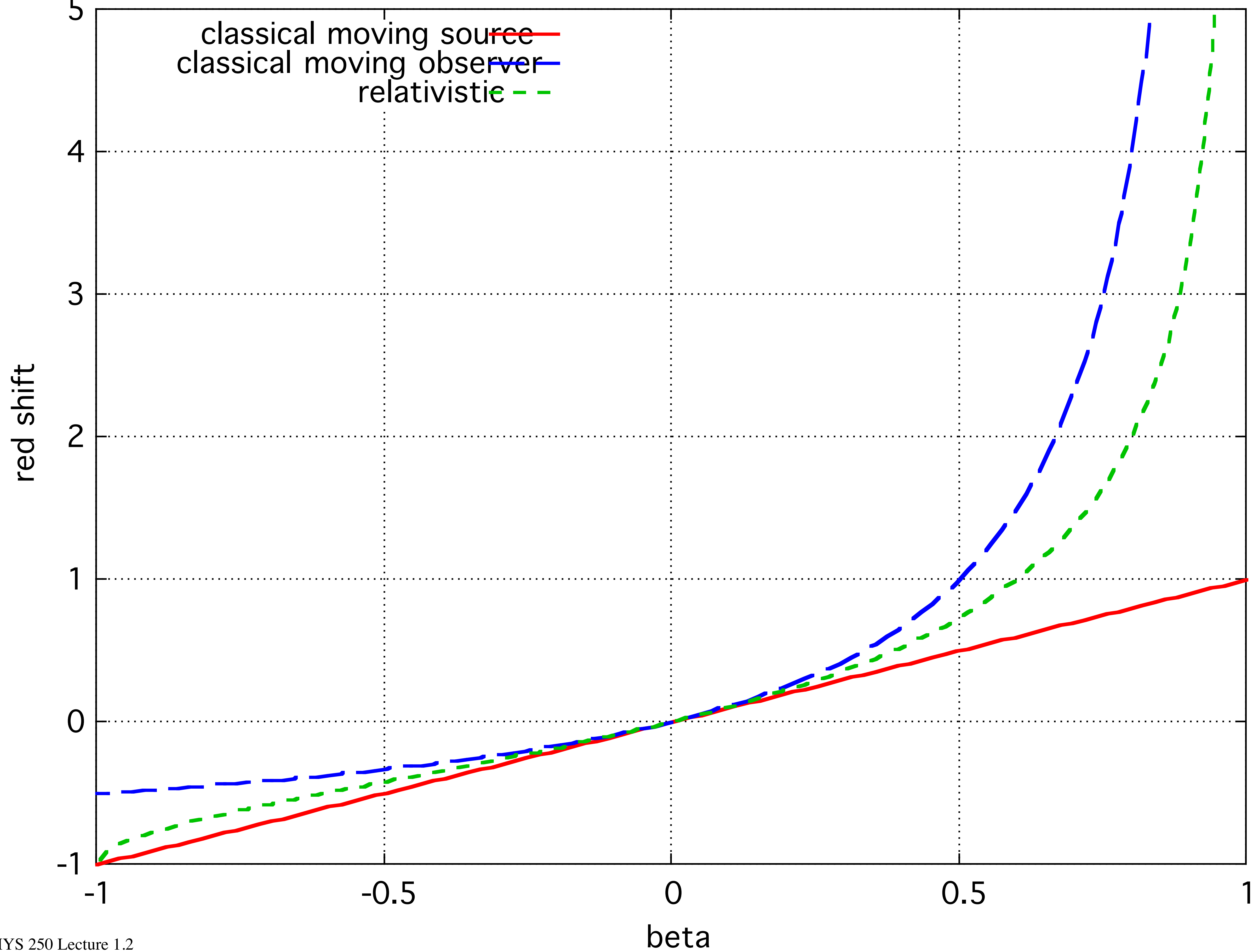
My convention: β is positive if the source and observer are <u>separating</u>.

Also be careful about period vs frequency vs wavelength.

Doppler Effect Comparison (Frequency)



Doppler Effect Comparison (Wavelength)



Time Dilation, Redshift, & Cosmology

Classically, a red shift greater than 1 means the source is moving away faster than the speed of light.

Relativity slows moving clocks down to zero as something approaches c.

So a distant galaxy with red shift larger than 1 isn't moving faster than light. The red shift is mostly time dilation.

Time dilation also means that we can still see the early Universe today! For parts of the Universe that are moving away from us fast enough, time is running so slowly that it's still just moments after the Big Bang.

That is why we can still see the cosmic background blackbody radiation (which was emitted very soon after the Big Bang, as ultraviolet light, but is red-shifted to the microwave region).

Classical Momentum and Velocity Addition

The classical momentum p is mass times velocity: $p = mv$.

If there are no external forces, like in a collision, total momentum is conserved:

$$\sum p_{\text{Before}} = \sum p_{\text{After}}$$

$$m_1 v_{1\text{Before}} + m_2 v_{2\text{Before}} = m_1 v_{1\text{After}} + m_2 v_{2\text{After}}$$

If the collision occurs in a frame with velocity u,

$$m_1\left(v_{1\text{Before}} + u\right) + m_2\left(v_{2\text{Before}} + u\right) = m_1\left(v_{1\text{After}} + u\right) + m_2\left(v_{2\text{After}} + u\right)$$

Collect the terms involving u on both sides:

$$m_1 v_{1\text{Before}} + m_2 v_{2\text{Before}} + \left(m_1 + m_2\right)u = m_1 v_{1\text{After}} + m_2 v_{2\text{After}} + \left(m_1 + m_2\right)u$$

The u terms cancel.

So if momentum is conserved in one frame, it's conserved in all frames.

Momentum Conservation and Relativity

The denominator in the relativistic velocity addition formula $v = \dfrac{v' + u}{1 + uv'/c^2}$

screws up the cancellation, which causes $p = mv$ momentum to <u>not</u> be conserved.

It's possible to invent a new definition of velocity,
and a new definition of momentum, and <u>that</u> momentum <u>is</u> conserved.

Imagine a clock that <u>moves along with each object</u>.
The time measured by such a clock is called <u>proper time</u> τ.

The normal velocity v is defined as $v = \dfrac{dx}{dt}$.

The <u>proper velocity w</u> is defined as $w = \dfrac{dx}{d\tau} = \dfrac{dx}{dt} \cdot \dfrac{dt}{d\tau} = v \cdot \dfrac{dt}{d\tau}$.

The relation between proper time τ and regular time t is $t = \gamma\tau$
so the relation between proper velocity and regular velocity is $w = v \cdot \gamma$

Relativistic Momentum *P*, and "*Q*"

Regular momentum is $p = mv = m\frac{d}{dt}x$.

Relativistic momentum is $P = mw = m\frac{d}{d\tau}x$.

We know that to transform *x* to a different frame,
we also need to transform *ct* at the different frame.

That implies that there should be some other quantity $Q = m\frac{d}{d\tau}ct$ involved when dealing with relativistic momentum.

The relation between regular time *t* and proper time τ is $t = \gamma\tau$.

So we can write $Q = m\frac{d}{d\tau}c\left(\gamma\tau\right) = mc\gamma = \frac{mc}{\sqrt{1 - v^2/c^2}}$.

Relativistic Energy

What is $Q = \dfrac{mc}{\sqrt{1 - v^2/c^2}}$?

Well, for $v << c$, the Taylor expansion is $Q \approx mc\left(1 + \dfrac{1}{2}\dfrac{v^2}{c^2}\right)$.

So $Qc = mc^2\left(1 + \dfrac{1}{2}\dfrac{v^2}{c^2}\right) = mc^2 + \dfrac{1}{2}mv^2$.

That's just mc^2 plus the classical kinetic energy $\text{K.E.} = \dfrac{1}{2}mv^2$.

So Qc is the relativistic energy total E.

Transforming Relativistic Momentum

Start with the Lorentz transform formulas, and do $m\frac{d}{d\tau}$ to all terms:

$$x' = \gamma\left(x - \beta ct\right) \qquad m\frac{d}{d\tau}x' = \gamma\left(m\frac{d}{d\tau}x - \beta m\frac{d}{d\tau}ct\right)$$

$$y' = y \qquad m\frac{d}{d\tau}y' = m\frac{d}{d\tau}y$$

$$z' = z \qquad m\frac{d}{d\tau}z' = m\frac{d}{d\tau}z$$

$$ct' = \gamma\left(ct - \beta x\right) \qquad m\frac{d}{d\tau}ct' = \gamma\left(m\frac{d}{d\tau}ct - \beta m\frac{d}{d\tau}x\right)$$

In terms of P and Q, this is just

$$P_x' = \gamma\left(P_x - \beta Q\right) = \gamma\left(P_x - \beta E/c\right)$$

$$P_y' = P_y$$

$$P_z' = P_z$$

$$Q' = \gamma\left(Q - \beta P_x\right) = \gamma\left(E/c - \beta P_x\right)$$

Relativistic Conservation

If we know relativistic momentum is conserved in one frame, $\sum \vec{P}_{\text{Before}} = \sum \vec{P}_{\text{After}}$ will it also be conserved in another frame?

For the *y* and *z* components, it's trivial. For the *x* component,

$$\sum P'_{x\text{Before}} = \sum \gamma \left(P_{x\text{Before}} - \beta Q_{\text{Before}} \right) = \gamma \sum P_{x\text{Before}} - \beta\gamma \sum Q_{\text{Before}}$$
$$\sum P'_{x\text{After}} = \sum \gamma \left(P_{x\text{After}} - \beta Q_{\text{After}} \right) = \gamma \sum P_{x\text{After}} - \beta\gamma \sum Q_{\text{After}}$$

So, $\sum \vec{P}'_{\text{Before}} = \sum \vec{P}'_{\text{After}}$ will be true if $\sum Q_{\text{Before}} = \sum Q_{\text{After}}$

is true as well as $\sum \vec{P}_{\text{Before}} = \sum \vec{P}_{\text{After}}$.

Since $Q = E/c$, $\sum Q_{\text{Before}} = \sum Q_{\text{After}}$ implies $\sum E_{\text{Before}} = \sum E_{\text{After}}$,
which is conservation of <u>relativistic total energy</u>.

Relativistic Conservation 2

In relativity, conservation of momentum and conservation of energy are tightly related.

In order for relativistic momentum to be conserved, relativistic total energy must also be conserved.

In order for relativistic total energy to be conserved, relativistic momentum must also be conserved.

Experimentally, both energy and momentum are conserved.

Calculating Momentum & Energy

Regular momentum is $p = mv$. If we know m and v, we just multiply.

Since $\beta = v/c$, regular $p = \beta mc$.

Relativistic momentum is $P = mw = m\dfrac{dx}{d\tau} = m \cdot \dfrac{dx}{dt} \cdot \dfrac{dt}{d\tau}$.

Plug in $\dfrac{dx}{dt} = v = \beta c$ and $\dfrac{dt}{d\tau} = \gamma$ to get $P = m \cdot \beta c \cdot \gamma = \beta\gamma mc$.

If we know v, we can find β, then γ, then multiply by mc.

Relativistic energy is similar: $E = Q \cdot c = mc\gamma \cdot c = \gamma mc^2$.

Given v or β, find γ, then multiply by mc^2.

But DO NOT use $E = \dfrac{1}{2}mv^2$ in relativity!!

Energy-Momentum-Mass Relation

We found that in relativity, $E = \gamma mc^2$ and $P = \beta\gamma mc$.

Plug those into the seemingly random expression $E^2 - \left(Pc\right)^2$
to get $\gamma^2 \cdot \left(mc^2\right)^2 - \left(\beta\gamma\right)^2 \cdot \left(mc^2\right)^2 = \left(\gamma^2 - \beta^2\gamma^2\right) \cdot \left(mc^2\right)^2$

Now expand: $\gamma^2 - \beta^2\gamma^2 = \gamma^2 \cdot \left(1-\beta^2\right) = \left(\dfrac{1}{\sqrt{1-\beta^2}}\right)^2 \cdot \left(1-\beta^2\right) = \dfrac{1-\beta^2}{1-\beta^2} = 1.$

That means $E^2 - \left(Pc\right)^2 = \left(mc^2\right)^2$, or equivalently $E^2 = \left(Pc\right)^2 + \left(mc^2\right)^2$

Every term has dimensions of energy-squared.

And if we could just get rid of all those factors of c…

Massless Particles

We can have $m = 0$ particles, as long as $E^2 = (pc)^2 + (mc^2)^2$ is still satisfied.

This implies $E^2 = (pc)^2 + 0$ or $E = pc$.

The photon (light, X-rays, gamma-rays) is exactly massless.
In c = 1 units, its energy equals its momentum.

Neutrinos are almost massless (sum of 3 types is 0.32 eV/c^2 from microwave background; difference ≈ 0.01 eV/c^2 from oscillation experiments).

Energy-Momentum-Mass Relation 2

We found that $E^2 = \left(Pc\right)^2 + \left(mc^2\right)^2$

Every term has dimensions of energy-squared.

And if we could just get rid of all those factors of c,

we would have the very simple $E^2 = P^2 + m^2 \ldots$

Electron-Volt Units

The energy scale of chemistry and atomic physics is the electron-Volt or eV: the energy change of an electron moving through a potential difference of one Volt.

Since the charge of an electron is 1.602×10^{-19} Coulombs, one eV is 1.602×10^{-19} Joules.

The ground state of hydrogen is –13.6 eV. The energy of inner shell electrons goes like Z^2, so for oxygen it's 870 eV (0.87 keV), and for uranium it's 1.15×10^{5} eV (115 keV or 0.115 MeV).

The convention in nuclear and particle physics is also to measure energies in eV, but the energies are millions, billions, or trillions of eV:

$$1\text{ MeV} = 10^{6}\text{ eV},\ \ 1\text{ GeV} = 10^{9}\text{ eV},\ \ \ 1\text{ TeV} = 10^{12}\text{ eV}$$

$c = 1$ Units

Energy has dimensions of mass times velocity squared.

Momentum has dimensions of mass times velocity,
which is the same dimensions as energy divided by velocity

And mass has the same dimensions as energy divided by velocity squared.

It's convenient to use energy in (multiples of) eletron-Volts,
and for the velocity we divide by, to be c, the speed of light.

Then a momentum has units of MeV/c or GeV/c or TeV/c.

And a mass has units of MeV/c^2 or GeV/c^2 or TeV/c^2.

The electron mass is 0.511 MeV/c^2 and the proton mass is 938.3 MeV/c^2

Clicker Question

What is the velocity, in m/s, of a proton with relativistic energy of 1200 MeV ?
(Proton mass is 938.3 MeV/c^2).

A. 1.87×10^8 m/s
B. 2.24×10^8 m/s
C. 3.00×10^8 m/s
D. 4.80×10^8 m/s
E. TE, DK (Too Early, Don't Know)

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$$E=\gamma mc^2 \;\rightarrow\; \gamma=\frac{E}{m\cdot c^2}=\frac{1200\text{ MeV}}{938.3\text{ MeV/c}^2\cdot c^2}=1.279$$

$$\beta=\sqrt{1-\gamma^{-2}}=0.6234 \;\rightarrow\; v=\beta c=0.6234\cdot 2.998\times10^8=1.869\times10^8\text{ m/s}$$

Dynamics in Relativity

You can't use $\vec{F} = m\vec{a}$ in relativity.

But the alternative form $\vec{F} = \frac{d\vec{P}}{dt}$ does work, using the relativistic momentum.

A possibly relativistic particle with charge q and ordinary velocity v moving in a uniform magnetic field B experiences Lorentz force $F = qvB$ and moves in a circle with radius R.

The P vector rotates at v/R radians per second, so $dP/dt = P \cdot v/R$.

Equating these, $F = qvB = P \cdot v/R \rightarrow P = qBR$.

The particle's relativistic momentum is proportional to the orbit radius.
The ordinary velocity cancels.

Particle physics detectors use this all the time to measure momenta, even when the velocity is unmeasurably close to the speed of light.

Dynamics in Relativity 2

If q, B, and R are in SI units, then $P = qBR$ will give a momentum, in SI units,

Multiply both sides by c to get $Pc = qcBR$, with both sides being energy in Joules

I'm assuming that the charge q of the particle is the electron charge.

Convert those to electron-Volts by dividing by 1.602×10^{-19} Joule/eV.

That gives $$(Pc)_{\text{eV}} = \frac{\left(q = 1.602\times 10^{-19}\right)\cdot\left(c = 2.997\times 10^{8}\right) B_{\text{Tesla}} R_{\text{meters}}}{1.602\times 10^{-19}\ \text{Joule/eV}}$$

or $(Pc)_{\text{eV}} = \left(2.997\times 10^{8}\ \text{eV/T-m}\right)\cdot B_{\text{Tesla}} R_{\text{meters}}$.

If $B = 1$ Tesla and $R = 1$ meter, fairly standard numbers,
then $Pc = 2.997\times 10^{8}\ \text{eV} = 299.7\ \text{MeV}$ so $P = 299.7\ \text{MeV/c}$.

Dynamics in Relativity 3

Another thing that works in relativity is $\Delta E = q \cdot \Delta V$
where ΔE is the relativistic energy change and ΔV is the voltage difference.

This is easier than the circular orbit case.

The energy change in electron-Volts is just the voltage difference in volts.

A van de Graaf generator with a voltage of 1 million volts can give a charge-1 particle an energy of 1 MeV.

For Next Time

Online homework (individual, no collaboration) for this week will be posted on Canvas tonight, due Monday midnight.

Friday tutorial will be doing a few problems in small groups, handed in electronically at Friday midnight.

Office Hours: 2:30-3:30 on Friday
Hennings 276, upstairs from the student machine shop.

Continuing at Koerner's….

No class Monday.

For Wednesday, read Y&F chapters
35.2 (two-slit interference)
39.5 (black body radiation)
38.1- (photo-electric effect, X-rays, Compton scattering)
36.6 (X-ray diffraction)