PHYS 250 Worksheet 4 Solutions

1. Electrons with kinetic energy of 9 eV are incident on a potential step from 0 V to 8 V located at $x = 0$. The incident electron beam has a density of 1 electron per micron (not cubic micron, because this is a one-dimensional problem).

A. What is the wavenumber k for the incident electrons?

$$E = \frac{\hbar^2 k^2}{2m} \rightarrow k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2mc^2}}{\hbar c} \cdot \sqrt{E} = \frac{\sqrt{2 \cdot 0.511 \times 10^6 \text{ eV}}}{197.3 \text{ eV-nm}} \cdot \sqrt{9 \text{ eV}} = \frac{5.124}{\text{nm-}\sqrt{\text{eV}}} \cdot 3\sqrt{\text{eV}} = 15.37 \text{ nm}^{-1}$$

B. What is the wavenumber k' for the transmitted electrons?

$$E - V = \frac{\hbar^2 k'^2}{2m} \rightarrow k' = \frac{\sqrt{2mc^2(E-V)}}{\hbar c} = \frac{5.124}{\text{nm-}\sqrt{\text{eV}}} \cdot \sqrt{(9-8) \text{ eV}} = 5.124 \text{ nm}^{-1}$$

C. What is the density of electrons in the reflected beam?

$$R = \frac{k - k'}{k + k'} = \frac{15.37 - 5.124}{15.37 + 5.124} = 0.5$$

This is the ratio of the amplitude of the reflected wave to the incident wave.
The probability density is the conjugate square of the amplitude, for either wave.
Since the incident wave has 1 electron per micron,
the reflected wave has 0.25 electrons per micron.

D. What is the density of electrons in the transmitted beam?

$$T = \frac{2k}{k + k'} = \frac{2 \cdot 15.37}{15.37 + 5.124} = 1.5$$

This is the ratio of the amplitude of the transmitted wave to the incident wave.
The probability density is the conjugate square of the amplitude, for either wave.
Since the incident wave has 1 electron per micron,
the transmitted wave has 2.25 electrons per micron.

E. What is the density of electrons at $x = 0$?

The wavefunction value must be continuous, so we should get the same value on either side of the step. So it must be $1.5^2 = 2.25 = (1 + 0.5)^2$ electrons per micron.

F. What is the minimum value of the electron density for $x < 0$?

$y_{\mathrm{I}}(x) = \exp(ikx), \quad y_{\mathrm{R}}(x) = R\exp(-ikx).$

The sum is $y_{\mathrm{I}}(x) + Ry_{\mathrm{R}}(x) = \exp(ikx) + R\exp(-ikx)$

The probability density is the conjugate square

$$\left[\exp(ikx) + R\exp(-ikx)\right]\cdot\left[\exp(-ikx) + R\exp(ikx)\right] = 1 + R\cdot\exp(2ikx) + R\cdot\exp(-2ikx) + R^2$$

$$= 1 + R^2 + 2R\cos(2kx)$$

The minimum value is when $\cos(2kx) = -1$, so it's $1 + R^2 - 2R = (1-R)^2 = 0.5^2 = 0.25$

So it's 0.25 electrons per micron.

G. What is the velocity of the incident electrons?

$$v = \frac{p}{m} = \frac{\hbar k}{m} = \frac{\hbar ck}{mc^2}c = \frac{197.4 \text{ eV-nm}\cdot 15.36 \text{ nm}^{-1}}{0.511\times 10^6 \text{ eV}}\cdot 2.998\times 10^8 \text{ m/s} = 1.779\times 10^6 \text{ m/s}$$

F. What is the incident electron flux (electrons/second) ?

$$\text{Flux} = \frac{1 \text{ electron}}{10^{-6} \text{ m}}\cdot 1.779\times 10^6 \text{ m/s} = 1.779\times 10^{12} \text{ electrons/second}$$

G. What is the reflected electron flux ?

It's $R^2 = 0.25$ times this, or 0.4447 x 10^{12} electrons/second.

H. What is velocity of the transmitted electrons?

$$v = \frac{\hbar ck'}{mc^2}c = \frac{197.4 \text{ eV-nm}\cdot 5.121 \text{ nm}^{-1}}{0.511\times 10^6 \text{ eV}}\cdot 2.998\times 10^8 \text{ m/s} = 0.5930\times 10^6 \text{ m/s}$$

I. What is the transmitted electron flux?

It's the transmitted density times the transmitted velocity:

$$\frac{2.25 \text{ electrons}}{10^{-6} \text{ m}}\cdot\frac{0.5930\times 10^6 \text{ m}}{\text{s}} = 1.334\times 10^{12} \text{ electrons/second}$$

Note that this is $(1.779 - 0.4447)\times 10^{12}$.

2. Using the simulation at https://phet.colorado.edu/en/simulation/legacy/quantum-tunneling select “barrier/well” for the potential, set the height “V2” to 1 eV, and set the width to 3 nm. You can do this by dragging on the potential graph, or through a dialog box. Leave the “Electron Average Total Energy” at the default 0.8 eV

A. Set the “Electron Wave Function” to “plane wave.” Describe in words what you see.

There is a standing wave upstream of the barrier, and a very small transmitted wave.

B. Change the “Electron Wave Function” to “wave packet” with the default “Initial Width” of 0.5 nm. How does the transmitted wave amplitude compare to case A?

There is a substantial transmitted wave. The initial height of the “Probability Density” Gaussian is about 1.1, and the transmitted Gaussian height is about 0.03.

C. Change the “Initial Width” of the wave packet to 3 nm, and the “Initial Position” to –8 nm. How does the transmitted wave amplitude compare to cases A and B ? How do you explain this?

The transmitted Gaussian is much smaller than B, and more comparable to the plane wave amplitude in A.

The narrow Gaussian has a large momentum spread and thus energy spread (the width of the green band in the top plot). So while the central energy of the wavefunction is less than the height of the barrier, and the barrier is wide enough that tunnelling is small, the part of the wavefunction with higher energy can still get through the barrier.

3. Quantum Harmonic Oscillator

A. What is $\hbar\omega$ in eV for an electron in a harmonic oscillator potential with $k = 1$ eV/nm^2 ?

$$\hbar\omega = \hbar\sqrt{\frac{k}{m}} = \hbar c\sqrt{\frac{k}{mc^2}} = \left(197.3\ \text{eV-nm}\right)\sqrt{\frac{1\ \text{eV/nm}^2}{0.511\times10^6\ \text{eV}}} = 0.2760\ \text{eV}$$

B. What is the scale factor b in nm for an electron in a harmonic oscillator potential with $k = 1$ eV/nm^2 ?

$$b^2 = \frac{\hbar}{\sqrt{km}} = \frac{\hbar c}{\sqrt{k\cdot mc^2}} = \frac{197.3\ \text{eV-nm}}{\sqrt{1\,\text{eV/nm}^2\cdot 0.511\times10^6\ \text{eV}}} = 0.2760\ \text{nm}^2$$

$$b = 0.5254\ \text{nm}$$

C. What is the energy in eV for an electron in the $n = 10$ state of a 1 dimensional harmonic oscillator potential with $k = 1$ eV/nm^2 ?

$$E = \left(n + \frac{1}{2}\right)\hbar\omega = \left(10 + \frac{1}{2}\right)\cdot 0.2760\ \text{eV} = 2.898\ \text{eV}$$

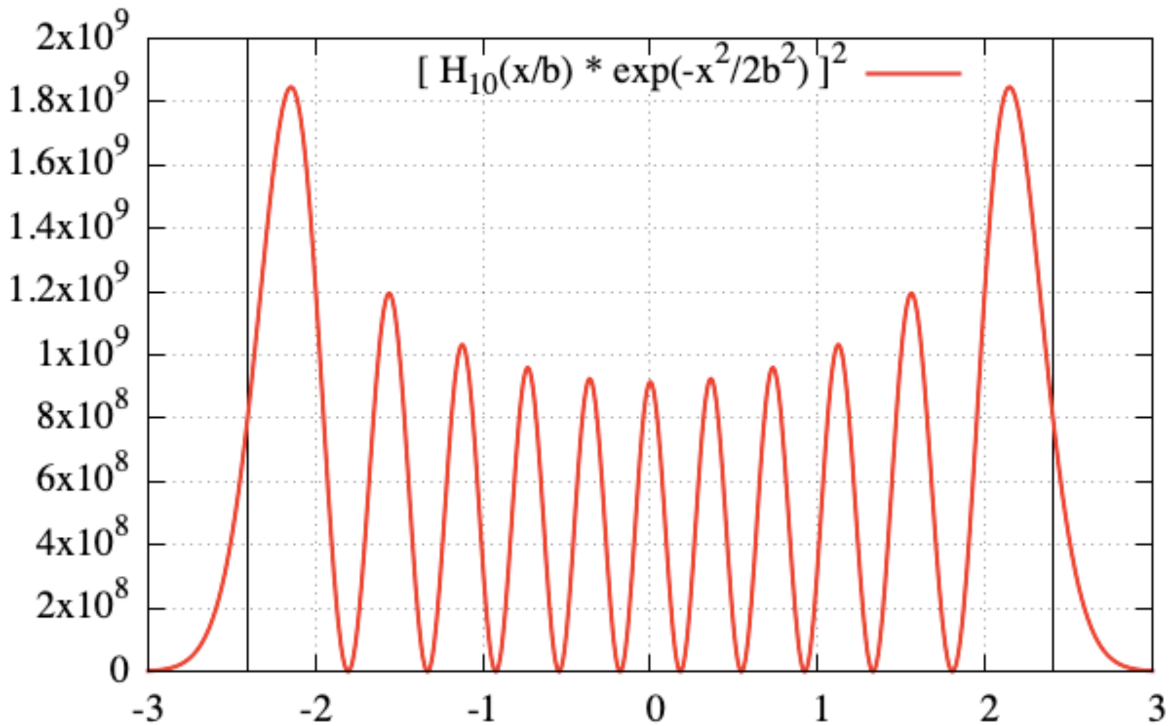
D. What is the classical amplitude A in nm for a particle in that potential with that energy?

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \quad\rightarrow\quad E = \frac{1}{2}kA^2 \quad\rightarrow\quad A = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2\cdot 2.898\ \text{eV}}{1\ \text{eV/nm}^2}} = 2.408\ \text{nm}$$

E. Make a graph of the probability distribution for an electron in the n = 10 state of a 1 dimensional harmonic oscillator potential with k = 1 eV/nm^2 . Don't worry about the normalization. Mark on the graph the extreme positive and negative excursions of a classical particle with that energy.

The probability distribution is the square of the wavefunction $\left[H_{10}\left(\frac{x}{b}\right)\cdot e^{\frac{-x^2}{2b^2}}\right]^2$

where $H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$



The quantum probability extends slightly beyond the classical turning points.