1) Spherical capacitor:

a) $\int \vec{D}\, d\vec{A} = 4\pi r^2 D = Q_{enc} = -Q$ for $\boxed{a<r<b}$

$$\Rightarrow \vec{D} = -\frac{Q}{4\pi r^2}\hat{r}$$

linear dielectric $\vec{E} = \frac{\vec{D}}{\varepsilon_0(1+\chi_e)} = \frac{-Q}{4\pi\varepsilon_0(1+\chi_e)r^2}\hat{r}$

and $\vec{P} = \varepsilon_0 \chi_e \vec{E} = -\frac{\chi_e}{1+\chi_e}\frac{Q}{4\pi r^2}\hat{r}$

$\vec{D} = \vec{E} = \vec{P} = 0$ for $r<a$ and $r>b$

b) bound charge: $\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} +\frac{\chi_e}{1+\chi_e}\frac{Q}{4\pi a^2} & \text{at } r=a \\ -\frac{\chi_e}{1+\chi_e}\frac{Q}{4\pi b^2} & \text{at } r=b \end{cases}$$

c) $V(b)-V(a) = -\int_a^b \vec{E}\cdot d\vec{l} = -\int_a^b -\frac{Q\,dr}{4\pi\varepsilon_0(1+\chi_e)r^2}$

$$= \frac{Q}{4\pi\varepsilon_0(1+\chi_e)}\left(\frac{1}{r}\right)\Big|_b^a = \frac{Q}{4\pi\varepsilon_0(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)$$

d) $C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0(1+\chi_e)}{\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\varepsilon_0(1+\chi_e)\,ab}{(b-a)}$ //

2. Polarized cylinder:

a) $\sigma_B = \vec{P}\cdot\hat{n} = \begin{cases} +k & \text{top surface} \\ -k & \text{bottom surface} \end{cases}$

$\operatorname{div}\vec{P} = 0 \Rightarrow \rho_b = 0$

b) $$V_{top}(z) = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{k\, 2\pi r\, dr}{\left(\left(z-\frac{a}{2}\right)^2 + r^2\right)^{1/2}} = \frac{k}{2\varepsilon_0}\left(\left(z-\frac{a}{2}\right)^2 + r^2\right)^{\frac{1}{2}}\Bigg|_0^R$$

$$= \frac{k}{2\varepsilon_0}\left[\left(\left(z-\frac{a}{2}\right)^2 + R^2\right)^{\frac{1}{2}} - z + \frac{a}{2}\right]$$

$$V_{bottom}(z) = \frac{-k}{2\varepsilon_0}\left[\left(\left(z+\frac{a}{2}\right)^2 + R^2\right)^{\frac{1}{2}} - z - \frac{a}{2}\right]$$

$V_{tot} = V_{top} + V_{bottom}$

c) this is a dipole, no net charge $\Rightarrow V_{dip} \sim \frac{1}{r^2} \sim \frac{1}{z^2}$

d) use Taylor expansion $\sqrt{1+\varepsilon} \approx 1 + \frac{\varepsilon}{2}$

$$V_{top} = \frac{k}{2\varepsilon_0}\left[\left(z-\frac{a}{2}\right)\sqrt{1 + \frac{R^2}{\left(z-\frac{a}{2}\right)^2}} - z + \frac{a}{2}\right]$$

$$\approx \frac{k}{2\varepsilon_0}\left[\left(z-\frac{a}{2}\right)\left(1 + \frac{1}{2}\frac{R^2}{\left(z-\frac{a}{2}\right)^2}\right) - z + \frac{a}{2}\right] = \frac{kR^2}{4\varepsilon_0}\frac{1}{z-\frac{a}{2}}$$

similarly $V_{bottom} \approx -\frac{kR^2}{4\varepsilon_0}\frac{1}{z+\frac{a}{2}}$

$$V_{tot} \approx \frac{kR^2}{4\varepsilon_0}\left(\frac{1}{z-\frac{a}{2}} - \frac{1}{z+\frac{a}{2}}\right) = \frac{kR^2}{4\varepsilon_0}\left(\frac{z+a/2 - z + a/2}{\left(z-\frac{a}{2}\right)\left(z+\frac{a}{2}\right)}\right) = \frac{kR^2 a}{4\varepsilon_0\left(z-\frac{a}{2}\right)\left(z+\frac{a}{2}\right)} \sim \frac{1}{z^2}$$

3) Wire:

a) $I = \int_0^R k s^3 \, 2\pi s \, ds = 2\pi k \frac{R^5}{5} \Rightarrow k = \frac{5I}{2\pi R^5}$ $\quad [k] = \left[\frac{A}{m^5}\right]$

b) inside $\oint \vec{B}\, d\vec{l} = \mu_0 \int_0^s k s'^3 \, 2\pi s' \, ds'$ or $\vec{B}_{in} = \frac{\mu_0 k s^4}{5} \hat{\phi}$

outside $\vec{B}_{out} = \frac{\mu_0 k R^5}{5 s} \hat{\phi}$

c) $B_\phi = -\frac{\partial A_z}{\partial s}$; $\quad$ inside $A_z(s) = -\frac{\mu_0 k}{25} (s^5 - R^5)$

to get $A_z = 0$ at $s = R$; $\quad A_x = A_y = 0$

d) $\vec{M} = \chi_m \vec{H}$ and $\vec{B} = \mu_0 (1 + \chi_m) \vec{H}$ and $\oint \vec{H}\, d\vec{l} = I_{free}$

inside $\vec{M} = \chi_m \frac{k s^4}{5} \hat{\phi}$ and $\vec{B} = \mu_0 (1 + \chi_m) \frac{k s^4}{5} \hat{\phi}$

outside no change; $\vec{M} = 0$

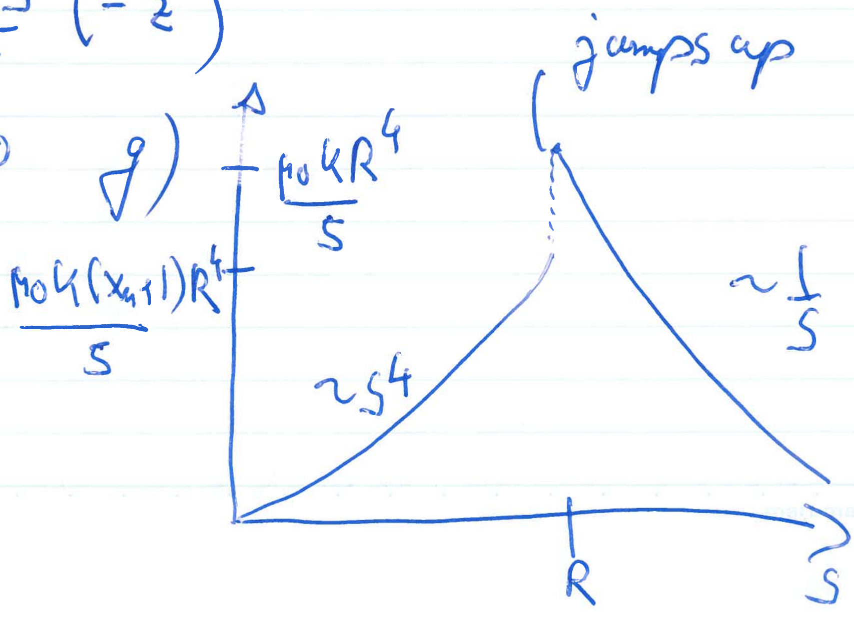
e) $\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{\chi_m}{s} \left[\frac{\partial}{\partial s}\left(s \frac{k s^4}{5}\right)\right] \hat{z} = \chi_m k s^3 \hat{z}$

$\vec{K}_b = \vec{M} \times \vec{n} = \chi_m \frac{k s^4}{5} (-\hat{z})$

f) diamagnetic: $\chi_m < 0$

Volume current down,
Surface current up,
$\vec{B}$-field is reduced
inside

g)



4) Capacitor:

a) By symmetry $\vec{B} = |\vec{B}_\phi| \hat{\phi}$

outside: $\oint \vec{B}\, d\vec{l} = 2\pi r\, B_\phi(r) \Rightarrow \vec{B}_{out} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$

inside: displacement current

$$I_d = \varepsilon_0 \int \frac{d\vec{E}}{dt} \cdot d\vec{A} = \varepsilon_0 \frac{d}{dt}\left(\frac{Q}{\varepsilon_0 \pi a^2}\right)\pi r^2 = \frac{I r^2}{a^2}$$

$$|\vec{E}| = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 \pi a^2}$$

$$\vec{B}_{in} = \frac{\mu_0 I r}{2\pi a^2}\hat{\phi}$$

Discontinuity: $\Delta \vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{r} - \frac{r}{a^2}\right)\hat{\phi}$

b) boundary condition $\vec{B}^{\parallel}_{out} - \vec{B}^{\parallel}_{in} = \mu_0 (\vec{K} \times \hat{n})$

for $\vec{K} = \hat{r}\frac{I}{2\pi}\left(\frac{1}{r} - \frac{r}{a^2}\right)$ we have

$$\mu_0 \vec{K} \times \hat{n} = \frac{\mu_0 I}{2\pi}\left(\frac{1}{r} - \frac{r}{a^2}\right)\hat{r} \times (-\hat{z})$$

note $\hat{r} \times \hat{z} = -\hat{\phi}$

$$= \frac{\mu_0 I}{2\pi}\left(\frac{1}{r} - \frac{r}{a^2}\right)\hat{\phi}$$

$$= \Delta \vec{B} \text{ as above}$$

c) annulus dr at r has area $dA = 2\pi r\, dr$

$$\frac{d}{dt}(dQ) = K(r)\, 2\pi r - K(r+dr)\, 2\pi (r+dr)$$

$$= K(r)\, 2\pi r - K(r+dr)\, 2\pi r - \underbrace{K(r+dr)\, 2\pi\, dr}_{= 2\pi K(r) dr + O(dr^2)}$$

neglect

$$= \frac{K(r) - K(r+dr)}{dr}\, dr\, 2\pi r - 2\pi K(r)\, dr$$

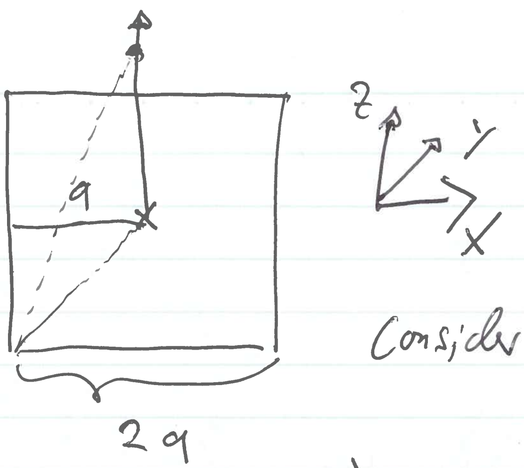
$$= -\frac{dK(r)}{dr}\, 2\pi r\, dr - 2\pi K(r)\, dr$$

$$\frac{d}{dt}\left(\frac{dQ}{2\pi r\, dr}\right) = \frac{d\sigma}{dt} = -\frac{dK}{dr} - \frac{K}{r} = \frac{I}{2\pi}\left(\frac{1}{r^2} + \frac{1}{a^2}\right) - \frac{I}{2\pi}\left(\frac{1}{r^2} - \frac{1}{a^2}\right)$$

$$= \frac{I}{\pi a^2}$$

is independent of position, so σ stays uniform

Square loop:



Consider one segment and use superpositi…

a) Biot-Savart: $$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \hat{\mathcal{R}}}{|\vec{\mathcal{R}}|^2}$$

$$\vec{\mathcal{R}} = \begin{pmatrix} +a \\ -y' \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} -a \\ y' \\ 0 \end{pmatrix} \qquad |\vec{\mathcal{R}}|^2 = a^2 + z^2 + y'^2$$

$$d\vec{l} = \begin{pmatrix} 0 \\ -dy' \\ 0 \end{pmatrix} \qquad d\vec{l} \times \hat{\mathcal{R}} = \frac{1}{|\vec{\mathcal{R}}|} \begin{pmatrix} 0 \\ -dy' \\ 0 \end{pmatrix} \times \begin{pmatrix} a \\ -y' \\ z \end{pmatrix} = \frac{1}{|\vec{\mathcal{R}}|} \begin{pmatrix} -dy\,z \\ 0 \\ a\,dy' \end{pmatrix}$$

counterclockwise current

By symmetry $B_x = B_y = 0$

$$B_z = \underbrace{4}_{4 \text{ segments}} \cdot \frac{\mu_0}{4\pi} I \int_{-a}^{+a} \frac{a\,dy'}{(z^2 + a^2 + y'^2)^{3/2}} = \frac{\mu_0 I a}{\pi} \left[\frac{y'}{(z^2+a^2)(z^2+a^2+y'^2)^{1/2}} \right]_{-a}^{+a}$$

$$= \frac{\mu_0 I a}{\pi} \frac{2a}{(z^2+a^2)(z^2+2a^2)^{\frac{1}{2}}}$$

b) Dipole moment $\vec{m} = I\,4a^2\,\hat{z}$

$$\vec{B} \approx \frac{\mu_0 m}{4\pi r^3} \left[2\hat{r}\cos\theta + \hat{\theta}\sin\theta \right] = \frac{\mu_0 2 I a^2}{\pi r^3} \hat{z}$$

we have $\theta = 0$, $\hat{r} = \hat{z}$

6) a)

Boundary conditions: $E_{\parallel}(P_1) = E_{\parallel}(P_2)$

$$E_{\perp}(P_2) - E_{\perp}(P_1) = \frac{\sigma}{\varepsilon_0}$$

So at P_2: $E_x(P_2) = E_x(P_1) = 4\,\frac{V}{c}$

$$E_z(P_2) = E_z(P_1) + \frac{\sigma}{\varepsilon_0} = 0 + 3\,\frac{V}{c}$$

$$\Rightarrow \vec{E}(P_2) = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \frac{V}{c}$$

b) from $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \oint \vec{B} \cdot d\vec{A} = 0$

c) $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

$\nabla^2 A_x = -\mu_0 J_x$ has same form as $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

Poisson-eq is solved by $V(r) = \int \frac{\rho \, d\tau}{|\vec{R}|}$

So replace $V \to A_x$

$\frac{1}{\varepsilon_0} \to \mu_0$

$\rho \to J_x$

$$A_x = \frac{\mu_0}{4\pi} \int \frac{J_x \, d\tau}{|\vec{R}|}$$