PHYS 301: Electricity and Magnetism
Term 1, 2022/23
FINAL EXAM, December 22, 2023
Time: 150 min

NAME: **Student Number:**

This is a closed book exam. There are **six** problems. Please write all answers directly into this exam. One handwritten formula sheet allowed. Calculators are allowed but not needed. Show all your work as partial credit will be given only if your reasoning is clear. Possibly useful formulae follow:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{R}|^2}\hat{\mathbf{R}} \tag{1}$$

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) \tag{2}$$

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} \tag{3}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{R}|} \tag{4}$$

$$W = \frac{1}{2}\int \rho V d\tau \tag{5}$$

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau \tag{6}$$

$$C = \frac{Q}{V} \qquad C = \frac{\epsilon_0 A}{d} \qquad W = \frac{1}{2}CV^2 \tag{7}$$

$$\nabla^2 V(\mathbf{r}) = -\rho/\epsilon_0 \tag{8}$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \tag{9}$$

$$\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E} = \epsilon_0(1+\chi_e)\mathbf{E} \quad \text{(linear dielectric)} \tag{10}$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \tag{11}$$

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{12}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{\text{quadrupole}}{r^3} + ...\right) \tag{13}$$

$$\mathbf{p} = \sum_i q_i\mathbf{r_i} \quad \text{(discrete charges)} \tag{14}$$

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau' \quad \text{(continuous charges)} \tag{15}$$

$$V(\mathbf{r}) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta)) \tag{16}$$

$$P_0(x) = 1, P_1(x) = x, P_2(x) = 0.5(3x^2-1) \tag{17}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\mathbf{R}}}{|\mathbf{R}|^2} \tag{18}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{19}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{R}|} d\tau' \tag{20}$$

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J} \tag{21}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{22}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 (1 + \chi_m) \mathbf{H} \quad \text{(linear magnetic material)} \tag{23}$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \tag{24}$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} \tag{25}$$

$$E_\perp^{above} - E_\perp^{below} = \sigma / \epsilon_0 \qquad \vec{E}_\parallel^{above} = \vec{E}_\parallel^{below} \tag{26}$$

$$\vec{B}_\perp^{above} - \vec{B}_\perp^{below} = \mu_0 \vec{K} \times \hat{n} \qquad B_\perp^{above} = B_\perp^{below} \tag{27}$$

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2} \tag{28}$$

$$\mathbf{m} = I \int d\mathbf{a}' \tag{29}$$

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{30}$$

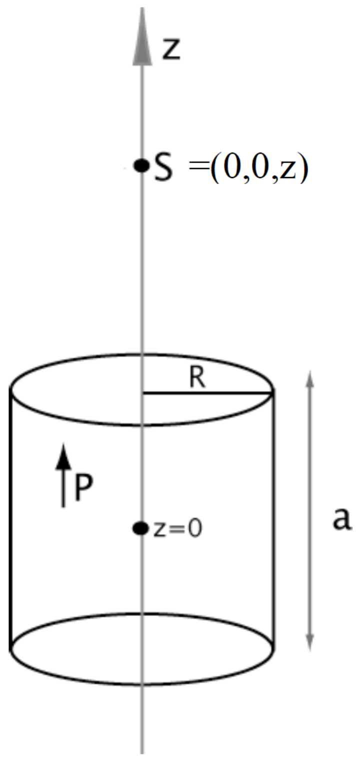
Question 1:
A spherical capacitor consists of an inner conducting spherical shell of radius a concentric with an outer conducting spherical shell of radius b. The volume in between the two shells is filled with a linear dielectric material with susceptibility χ_e.
(a) [4 pts] A total charge $-Q$ is placed on the inner condutor and a total charge of $+Q$ is placed on the outer conductor. Find the electric field $\vec{E}$, the displacement field $\vec{D}$, and the polarization $\vec{P}$ in the volume between the conducting shells.

(b) [1 pts] Find all bound charges in the capacitor

(c) [2 pts] Find the magnitude of the voltage difference between the inner and outer conductors. Is this larger or smaller compared to the case of no dielectric material between the shells?

(d) [1 pts] Find the capacitance C.

Question 2:
A solid dielectric cylinder has a uniform frozen-in polarization of $\vec{P} = k\hat{z}$, where k is a constant. The height of the cylinder is a, and the radius of the cylinder is R, as in the figure. The cylinder is centered on the origin (so e.g. the top surface is at $z = +a/2$).



(a) [1 pts] Calculate all bound charges. If any are zero, state clearly why.

(b) [5 pts] From your answer above, calculate the **potential** outside the cylinder at point S, on the z-axis with $z > 0$, by direct integration.

(c) [2 pts] How would you expect the potential to drop off with r (and thus z) for large distances? Or, what is the leading term in the multipole expansion for this configuration, and what power of $1/r$ does it depend on? Please answer even if you did not complete part (b).

(d) [4 pts] Now show that your answer from part (b) gives that z-dependence for large z, i.e. $z \pm a/2 \gg R$.

Question 3:
A steady current I flows up an infinitely long cylindrical wire of radius R. The current density in the wire is NOT uniform, it is given by $J(s) = ks^3$.
(a) [1 pts] Determine the constant k. (Including units!)

(b) [2 pts] Calculate $B(s)$ inside and outside the wire. Clearly indicate direction and magnitude.

(c) [2 pts] Find the vector potential $\mathbf{A}(s)$ inside the wire if $A = 0$ at $s = R$.

(d) [2 pts] Redo part (b), now assuming the wire has a uniform nonzero magnetic susceptibility χ_m.

(e) [2 pts] Given part (d), find all bound currents everywhere in (and on) the wire.

(f) [1 pts] Now assume the wire in parts (d) and (e) is diamagnetic. Describe which direction the bound currents (both surface and volume) flow, and then explain in words, how the direction(s) make sense, given what you know about diamagnetic materials.

(g) [2 pts] For the (diamagnetic) case of part (f), sketch a graph of $|B(s)|$ (the magnitude of the B field, as a function of s, both inside and outside). In particular, discuss if your $B(s)$ is continuous or not at $s = R$, and why?

Question 4:
A parallel plate capacitor with circular plates of radius a is charging with current I.
(a) [4 pts] Find the magnetic field $\vec{B}$ in between the plates (i.e. for $r \leq a$) and outside of the plates around the wire connected to the middle of the plates, and from that compute the discontinuity of the magnetic field across the capacitor plates.

(b) [2 pts] With the result from (a), show that the surface current density that flows radially outward as the plates become charged can be written as

$$\mathbf{K} = \frac{I}{2\pi}\left(\frac{1}{r} - \frac{r}{a^2}\right)\hat{r}.$$

Here $\hat{r}$ is the unit vector in the radial direction from the centre of the plate where the current is introduced.

(c) [4 pts] Despite the flowing surface current, show that the charge remains uniformly distributed over the area of the plates during charging. (Hint: Cosider the rate by which a charge dQ flows into an annulus of radius r and thickness dr, to first order in dr. Then look for an expression for the rate by which the surface charge density changes.)

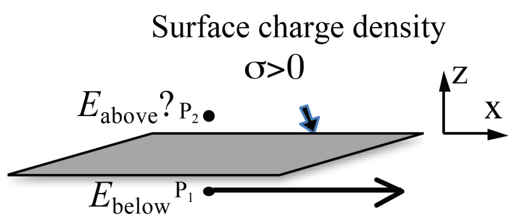
Question 5:
A square wire loop of size $2a \times 2a$ lies in the xy-plance with its centre at the origin and sides parallel to the x and y axes. A counterclockwise current I runs around the loop.
(a) [6 pts] Find the magnetic field on the z-axis. The following integrals may be useful:

$$\int \frac{dx}{(a^2 - x^2)^{1/2}} = \arcsin(x/a) \qquad \int \frac{dx}{x^2(a^2 - x^2)^{1/2}} = -\frac{(a^2 - x^2)^{1/2}}{a^2 x}$$

$$\int x(a^2 + x^2)^{1/2} dx = \frac{1}{3}(a^2 + x^2)^{3/2} \qquad \int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2(a^2 + x^2)^{1/2}}$$

(b) [2 pts] Find an approximation of the B-field for $z \gg a$. The leading order contribution suffices. You can answer even if you did not complete the calculation in part (a).

Short Questions:
(a) [3 pts] You have a sheet with a uniform (positive) surface charge density σ. The sheet lies in the xy-plane (perpendicular to the plane of this page, it is shown in perspective). There are lots of OTHER charges just out of the picture (not shown!) contributing to the E field.



At a point P_1 just (infinitesimally) below this sheet, the electric field is $\vec{E} = E_0\hat{x}$. The numerical values of σ and E_0 are given by $\sigma/\epsilon_0 = 3N/C$ and $E_0 = 4N/C$. What is the electric field at point P_2 just above the sheet (infinitestimally above point P_1)?. Either specify all components or magnitude and direction.

(b) [2 pts] A long straight wire, carrying current I, is aligned with the z-axis. How do we know that the magnetic field has no radial component in cylindrical coordinates, that is, no component pointing directly towards or away from the z-axis?

(c) [2 pts] In the Coulomb gauge, the vector potential can be determined from the equation $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$, where $\mathbf{J}$ is the current density. Why can we immediately say that the solution to this differential equation is $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{R}|}d\tau'$?

Extra space: