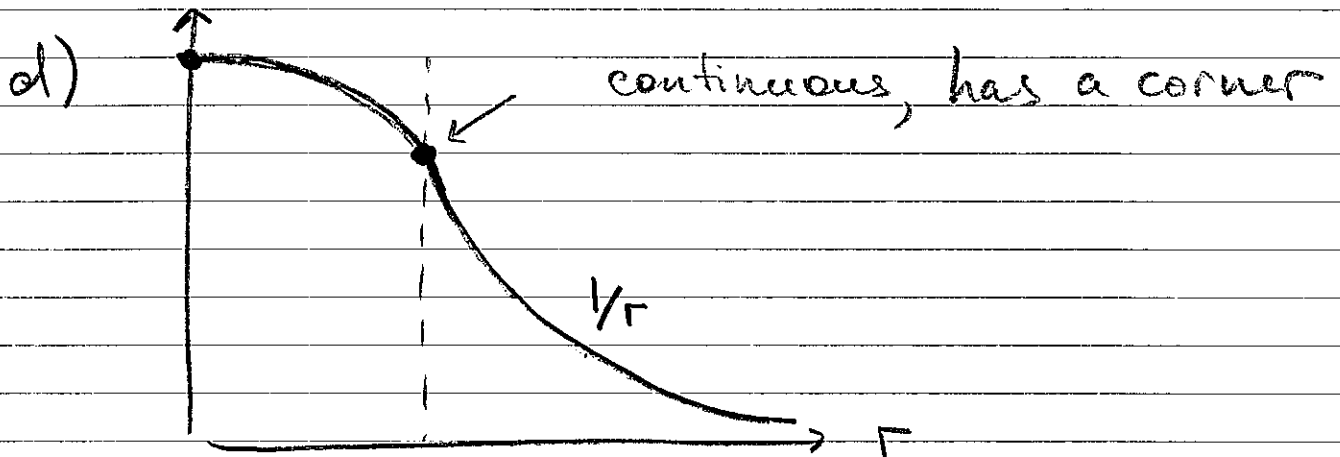


Final exam PHYS301 2024

Problem 1

- a) $\rho(r)$: positive ($E_r > 0$), uniform ($E_r = k \cdot r$)
- b) $\sigma(r=R)$: positive ($E \uparrow$), uniform ($E_\theta = E_\phi = 0$)
- c) It's an insulator (for conductor, $E_{in} = 0$)

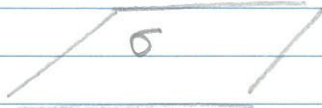


Problem 2

a) Superposition principle

b)

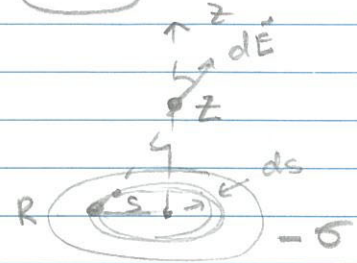
System =



+



$$\vec{E}_{\text{plane}}(z) = \frac{\sigma}{2\epsilon_0} \hat{z}$$



$$\vec{E}_{\text{disk}} = \int d\vec{E}_{\text{ring}};$$

$$\vec{E}_{\text{ring}} = \int d\vec{E}_q = \int_{\text{ring}} \frac{1}{4\pi\epsilon_0} \frac{dq}{(z^2 + s^2)^{3/2}} \vec{z} =$$

$$= \frac{1}{4\pi\epsilon_0} \frac{z q}{(z^2 + s^2)^{3/2}}; \quad q \rightarrow \sigma \cdot 2\pi s ds$$

$$\vec{E}_{\text{disk}} = \int_0^R \frac{1}{4\pi\epsilon_0} \frac{\sigma \cdot 2\pi s ds \cdot \vec{z}}{(z^2 + s^2)^{3/2}} = \frac{\sigma}{2\epsilon_0} \vec{z} \int_0^R \frac{ds \cdot s}{(z^2 + s^2)^{3/2}} =$$

$$= -\frac{\sigma}{2\epsilon_0} \vec{z} \cdot \frac{1}{\sqrt{s^2 + z^2}} \Big|_0^R = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right) \vec{z}$$

$$\vec{E}_{\text{disk}}(z) = -\frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right) \hat{z}$$

$$\vec{E}(z) = \frac{\sigma}{2\epsilon_0} \frac{z}{\sqrt{R^2 + z^2}} \cdot \hat{z}$$

$$\begin{aligned}
 c) \quad \Delta V &= V(z) - V(0) = - \int_0^z \frac{\sigma}{2\epsilon_0} \frac{z'}{\sqrt{R^2 + (z')^2}} dz' = \\
 &= - \frac{\sigma}{2\epsilon_0} \cdot \sqrt{R^2 + (z')^2} \Big|_0^z \Rightarrow
 \end{aligned}$$

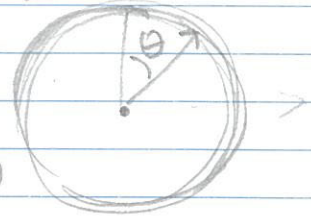
$$\boxed{V(z) = - \frac{\sigma}{2\epsilon_0} (\sqrt{z^2 + R^2} - R)}$$

$$d) \text{ If } z \gg R: \quad \boxed{V(z) \approx - \frac{\sigma}{2\epsilon_0} z} \quad \text{like for a plane}$$

e) No, we can't: we cannot be "very far away" from an infinite plane.

Problem 3

$$\sigma(\theta) = \sigma_0 \cos \theta$$



$$a) \quad V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$V_{in}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \quad \left| \begin{array}{l} (r^{-(l+1)})' \\ = -\frac{(l+1)}{r^{l+2}} \end{array} \right.$$

$$V_{out}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

Continuity at R:

$$\sum_l A_l R^l P_l(\cos \theta) = \sum_l \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

$P_l(\cos \theta)$ are orthogonal $\rightarrow$

$$A_l R^l = \frac{B_l}{R^{l+1}} \rightarrow \boxed{B_l = A_l R^{2l+1}}$$

Matching derivative:

$$-\frac{\partial V_{out}}{\partial r} + \frac{\partial V_{in}}{\partial r} = \frac{\sigma_0}{\epsilon_0} \cos \theta$$

$$\frac{\partial V_{out}}{\partial r} = \sum_{l=0}^{\infty} -\frac{B_l (l+1)}{r^{l+2}} P_l(\cos \theta)$$

$$\frac{\partial V_{in}}{\partial r} = \sum_{l=0}^{\infty} A_l \cdot l \cdot r^{l-1} \cdot P_l(\cos \theta)$$

$l=1$ Hence:

$$\left(\sum_{l=0}^{\infty} \left(\frac{B_l (l+1)}{r^{l+2}} + A_l \cdot l \cdot r^{l-1} \right) P_l(\cos \theta) \right) \bigg|_{r=R} = \frac{\sigma_0}{\epsilon_0} P_1(\cos \theta)$$

From orthogonality of Legendre polynomials:

$$\frac{2B_1}{R^3} + A_1 = \frac{\sigma_0}{\epsilon_0}, \quad \text{all other coefficients are equal to zero}$$

$$\cancel{\frac{2}{R^3}} \cdot A_1 R^3 + A_1 = \frac{\sigma_0}{\epsilon_0} \rightarrow A_1 = \frac{\sigma_0}{3\epsilon_0} \rightarrow B_1 = \frac{\sigma_0}{3\epsilon_0} R^3$$

Hence:

$$\boxed{\begin{aligned} V_{in}(r, \theta) &= \frac{\sigma_0 r}{3\epsilon_0} \cos \theta \\ V_{out}(r, \theta) &= \frac{\sigma_0}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta \end{aligned}}$$

b) $Q=0$ since total charge on the sphere is zero.

$$\vec{p} = \int \vec{r}' \cdot \rho(r') d\tau' \rightarrow \int \vec{r}' \sigma(\theta) da'$$

$p_x = p_y = 0$ by symmetry

$$p_z = \int (R \cos \theta) \cdot (\sigma_0 \cos \theta) (R \sin \theta d\phi \cdot R d\theta) =$$

$$= \sigma_0 R^3 \cdot 2\pi \int_0^\pi \sin\theta \cdot \cos^2\theta \cdot d\theta = \sigma_0 R^3 2\pi \cdot \frac{2}{3} \rightarrow$$

$$\vec{p} = \frac{4\pi R^3}{3} \sigma_0 \hat{z} \rightarrow$$

$$\hat{z} \cdot \hat{r} = \cos\theta$$

$$V^{(1)}(r, \theta) = \frac{\hat{r} \cdot \vec{p}}{4\pi\epsilon_0 r^2} =$$

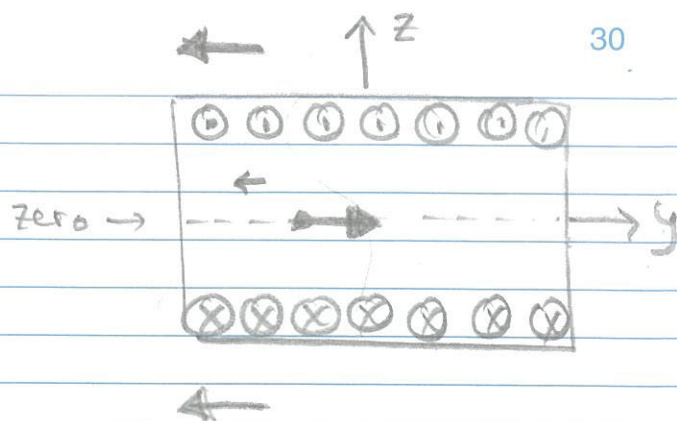
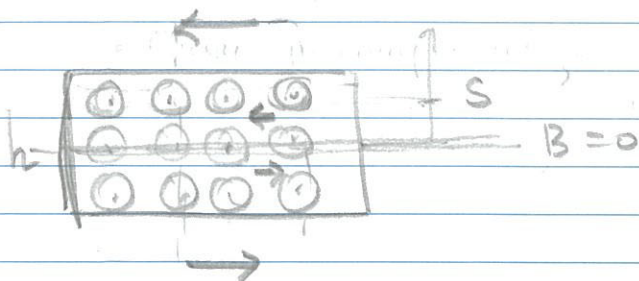
$$= \frac{4\pi R^3}{3} \sigma_0 \cdot \frac{\cos\theta}{4\pi\epsilon_0 r^2} \rightarrow$$

$$\boxed{V^{(1)}(r, \theta) = \frac{\sigma_0}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta}$$

same as
before!!

Problem 4

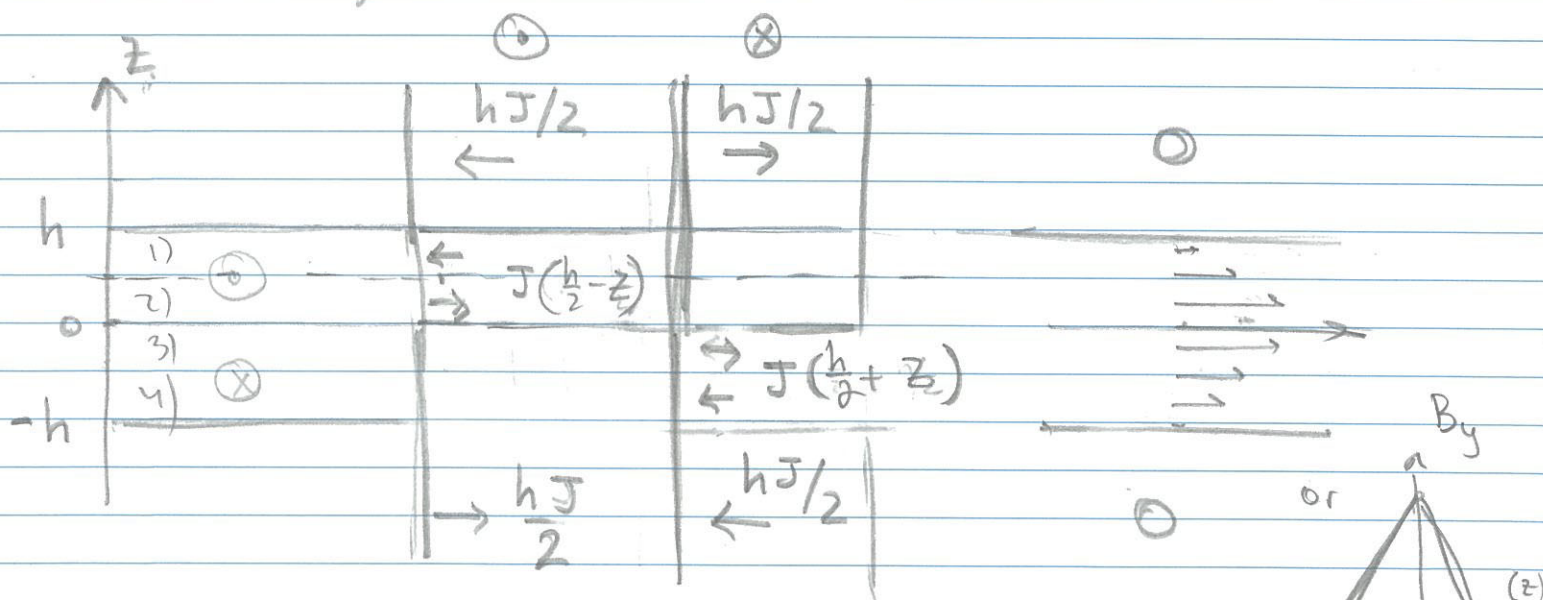
Upper slab:



Outside: $2B\ell = \ell \cdot h \cdot J \rightarrow B = \frac{hJ}{2}$

Inside: $2B\ell = \ell \cdot 2s \cdot J \rightarrow B = sJ$

For negative slab - the other way around



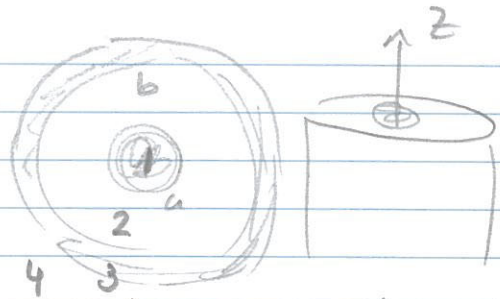
$$0 < z < h: J\left(\frac{h}{2} - z\right) + J\frac{h}{2} = J(h - z)$$

$$-h < z < 0: J\frac{h}{2} + J\left(\frac{h}{2} + z\right) = J(h + z)$$

Problem 5

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a) Magnetization:



$\vec{M} = \chi_m \vec{H}$; - exists only inside magnetic ($a < s < b$)

There: $H_\phi \cdot 2\pi s = I \rightarrow H_\phi = I / 2\pi s \rightarrow$

$$\boxed{\vec{M}_2 = \frac{\chi_m I}{2\pi s} \hat{\phi} \rightarrow}$$

$$\vec{J}_b = \nabla \times \vec{M} = \frac{\chi_m I}{2\pi} \frac{1}{s} \frac{\partial}{\partial s} \left(s \cdot \frac{1}{s} \right) \hat{z} = 0$$

$$\boxed{J_b = 0} \text{ (linear magnetic)} \quad \boxed{J_b = 0 \text{ everywhere}}$$

b) $\vec{K}_B = \vec{M} \times \hat{n}$

$s=a: \hat{n} = -\hat{s}, \hat{\phi} \times (-\hat{s}) = \hat{z}$

$s=b: \hat{n} = \hat{s}, \hat{\phi} \times \hat{s} = -\hat{z}$

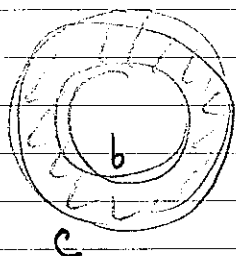
and they are zero at $s=c$.

$$\boxed{\begin{aligned} \vec{K}_a &= \frac{\chi_m I}{2\pi a} \hat{z} \\ \vec{K}_b &= \frac{\chi_m I}{2\pi b} (-\hat{z}) \end{aligned}}$$

c) $s < a: H \cdot 2\pi s = \frac{s^2}{a^2} I \rightarrow \boxed{H_\phi = \frac{sI}{2\pi a^2}, B_\phi = \frac{\mu_0 s I}{2\pi a^2}}$

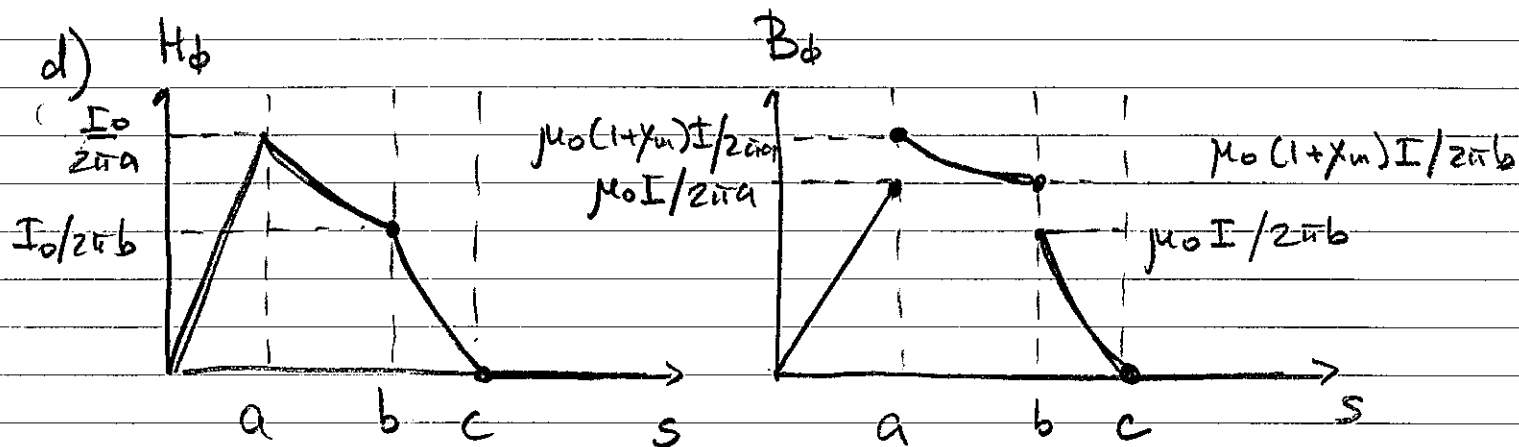
$a < s < b: H \cdot 2\pi s = I \rightarrow \boxed{H_\phi = \frac{I}{2\pi s}, B_\phi = \frac{\mu_0 (1 + \chi_m) I}{2\pi s}}$

$$b < s < c: H \cdot 2\pi s = I - I \frac{s^2 - b^2}{c^2 - b^2} = I \frac{c^2 - b^2 - s^2 + b^2}{c^2 - b^2}$$



$$H_\phi = \frac{I}{2\pi s} \frac{c^2 - s^2}{c^2 - b^2} \quad B_\phi = \frac{\mu_0 I}{2\pi s} \frac{c^2 - s^2}{c^2 - b^2}$$

$$s > c: H_\phi = 0, B_\phi = 0 \quad \text{Zero net current enclosed}$$



$$e) H_{||}^{(1)} - H_{||}^{(2)} = K_f \equiv 0 \rightarrow \boxed{H \text{ continuous at both } a \text{ and } b}$$

↑ we only have volume currents

$$B_{||}^{(2)} - B_{||}^{(1)} = \mu_0 K_\perp$$

$$\text{at } s=a: \Delta B_\phi = \frac{\mu_0 \chi_m I}{2\pi a}, \quad K_{a,z} = \frac{\chi_m I}{2\pi a} \quad \text{correct!}$$

$$\text{at } s=b: \Delta B_\phi = -\frac{\mu_0 \chi_m I}{2\pi b}, \quad K_{b,z} = -\frac{\chi_m I}{2\pi b} \quad \text{correct!}$$

Problem 6.

a) • I_{ind} CCW $\rightarrow B_{ind} \odot$

• Area is decreasing $\rightarrow \Phi_m$ is decreasing
 $\rightarrow \vec{B}_{ind} \uparrow \uparrow \vec{B}_{ext} \rightarrow$

$B_{ext} \odot$

b) "Force definition", Assuming positive charges (conventional current):

$\vec{F}_L$ towards the edge $\rightarrow$ creates current,

as per $\mathcal{E} = \oint_{loop} \vec{f}_L \cdot d\vec{e}$

